

Homework 5: Due Wednesday, March 1

Problem 1: Determine both the left cosets and the right cosets of the subgroup H of the given group G in each of the following cases (make sure you completely determine H first!).

a. $G = D_4$ and $H = \langle r^2 s \rangle$.

b. $G = A_4$ and $H = \langle (1\ 2\ 3) \rangle$.

Hint: Save as much work as you can by using the general fact that you are working with equivalence classes of a certain equivalence relation, and you know that the equivalence classes partition G .

Problem 2: Suppose that G and H are finite groups. Show that if $|G|$ and $|H|$ are not relatively prime, then $G \times H$ is not cyclic (regardless of whether G and H are cyclic).

Note: This is the converse to Problem 7c on Homework 4.

Problem 3: Let G be a group and let $H = \{(a, a) : a \in G\}$.

a. Show that H is a subgroup of $G \times G$.

b. Assuming G is a finite group with $|G| = n$, compute $[G \times G : H]$.

Problem 4: Let G be a group. Suppose that H and K are finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e\}$.

Hint: Make use of Lagrange's Theorem.

Problem 5: Let H be a subgroup of G and let $a \in G$. Show that if $aH = Hb$ for some $b \in G$, then $aH = Ha$. In other words, if the left coset aH equals *some* right coset of H in G , then it must equal the right coset Ha .

Hint: Use the general theory of equivalence relations to simplify your life.

Problem 6: Suppose that H is a subgroup of a group G with $[G : H] = 2$. Suppose that $a, b \in G$ with both $a \notin H$ and $b \notin H$. Show that $ab \in H$.

Hint: Think about the four cosets eH, aH, bH , and abH .

Problem 7: Consider the group $(\mathbb{Q}, +)$. Notice that \mathbb{Z} is a subgroup of \mathbb{Q} , and in fact it is a normal subgroup of \mathbb{Q} because \mathbb{Q} is abelian. Thus, we can form the quotient \mathbb{Q}/\mathbb{Z} . In class, we mentioned that for every $q \in \mathbb{Q}$, there exists $r \in \mathbb{Q}$ with $0 \leq r < 1$ such that $q + \mathbb{Z} = r + \mathbb{Z}$. For example, we have $\frac{5}{2} + \mathbb{Z} = \frac{1}{2} + \mathbb{Z}$ and $-\frac{1}{7} + \mathbb{Z} = \frac{6}{7} + \mathbb{Z}$. In other words, we have

$$\mathbb{Q}/\mathbb{Z} = \{r + \mathbb{Z} : r \in \mathbb{Q} \cap [0, 1)\}.$$

a. Show that if $r_1, r_2 \in \mathbb{Q}$ with $0 \leq r_1 < r_2 < 1$ then $r_1 + \mathbb{Z} \neq r_2 + \mathbb{Z}$.

b. Show that every element of \mathbb{Q}/\mathbb{Z} has finite order.

Note: Thus, \mathbb{Q}/\mathbb{Z} is an infinite abelian group in which every element has finite order.