Homework 4: Due Wednesday, February 22

Problem 1: Find the order of the following elements in the given direct product:
a. $((1\ 6\ 4)(3\ 7), (1\ 4\ 2\ 3)) \in S_9 \times S_4$.
b. $(\overline{5}, \overline{7}, \overline{44}) \in \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/84\mathbb{Z}$.
c. $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \overline{3}) \in GL_2(\mathbb{R}) \times U(\mathbb{Z}/13\mathbb{Z})$.

Problem 2: Let G be group and let $a, g \in G$. The element gag^{-1} is called a *conjugate* of a.
a. Show that $(gag^{-1})^n = ga^ng^{-1}$ for all $n \in \mathbb{Z}$. You should start by giving a careful inductive argument for $n \in \mathbb{N}$.
b. Show that $|gag^{-1}| = |a|$. Thus, every conjugate of a has the same order as a.

Problem 3: Let G be a group, and assume that every element of G has finite order. Let $H \subseteq G$, and suppose that $e \in H$ and that H is closed under the group operation. Show that H is a subgroup of G.
Hint: Every finite group satisfies the hypothesis that every element has finite order (by Corollary 5.2.4).

Problem 4: Let $n \in \mathbb{N}$ with $n \geq 2$.
a. Show that $\{(1\ \ a) : 2 \leq a \leq n\}$ generates S_n.
b. Show that $\{(a\ \ a+1) : 1 \leq a \leq n-1\}$ generates S_n.
c. Show that $\{(1\ \ 2), (1\ \ 2\ \ 3\ \cdots\ n)\}$ generates S_n.
Hint: Don't reinvent the wheel every time. You already know you can get everything from the transpositions. Once you've done part (a), you know you can get everything from that smaller set, etc.

Problem 5: Let $n \geq 3$. Working in D_n, determine $|r^k s^\ell|$ for each $k, \ell \in \mathbb{N}$ with $0 \leq k \leq n-1$ and $0 \leq \ell \leq 1$.

Problem 6: Let $n \geq 3$.
a. Show that if $a \in D_n$ and $a \in \langle r \rangle$, then $sa = a^{-1}s$.
b. Show that if $a \in D_n$ but $a \notin \langle r \rangle$, then $ra = ar^{-1}$.
c. Find $Z(D_n)$. Your answer will depend on whether n is even or odd.
Hint for (c): Start by use parts (a) and (b) to argue that almost all elements of D_n are *not* in $Z(D_n)$.

Problem 7: Suppose that G and H are groups.
a. Show that if $G \times H$ is cyclic, then both G and H are cyclic.
b. Give a counterexample to the following statement: If G and H are both cyclic, then $G \times H$ is cyclic.
c. Suppose that G and H are both finite and cyclic. Assume also that $|G|$ and $|H|$ are relatively prime. Show that $G \times H$ is cyclic.