Homework 2: Due Wednesday, February 8

Problem 1: Let $a, b, c \in \mathbb{Z}$. Suppose that $5 \nmid a$, that $a \mid 7b + 2c$, and that $a \mid 8b + 3c$. Show that $a \mid b$.

Problem 2: Let $a, b, c \in \mathbb{Z}$ with $a \geq 0$. Show that $\gcd(ab, ac) = a \cdot \gcd(b, c)$.
Aside: Think about why we have the assumption that $a \geq 0$ here. What would happen if $a < 0$?
Hint: Let $m = \gcd(b, c)$. Show directly that am satisfies the defining properties of $\gcd(ab, ac)$.

Problem 3: In this problem, we show how to use the existence of the gcd of two integers to prove the existence of the gcd of three integers (which can, of course, be further generalized). Let $a, b, c \in \mathbb{Z}$, and let $m = \gcd(a, \gcd(b, c))$.
a. Show that m divides each of a, b, and c.
b. Show that if d divides each of a, b, and c, then $d \mid m$.
c. Show that there exist $x, y, z \in \mathbb{Z}$ with $ax + by + cz = m$.

Problem 4: Let $p \in \mathbb{N}^+$ be prime. Define a function $\mathsf{ord}_p \colon \mathbb{Z}\backslash\{0\} \to \mathbb{N}$ as follows. Given $a \in \mathbb{Z}\backslash\{0\}$, let $\mathsf{ord}_p(a)$ be the largest $k \in \mathbb{N}$ such that $p^k \mid a$. For example, we have $\mathsf{ord}_3(45) = 2$ and $\mathsf{ord}_3(10) = 0$. Without using the Fundamental Theorem of Arithmetic, prove that for all $p, a, b \in \mathbb{Z}\backslash\{0\}$ with p prime, we have $\mathsf{ord}_p(ab) = \mathsf{ord}_p(a) + \mathsf{ord}_p(b)$.

Problem 5: Let $A = \mathbb{N}^+$ and define $a \sim b$ to mean that there exists $n \in \mathbb{Z}$ with $a = 2^n b$.
a. Show that $\sim$ is an equivalence relation on A.
b. Characterize (with proof) which elements of $\mathbb{N}^+$ are the smallest elements of their equivalence class. In other words, find a simple characterization of the set $\{a \in \mathbb{N}^+ : a \leq b \text{ for all } b \in \mathbb{N}^+ \text{ with } a \sim b\}$.

Problem 6: Let A, B, C be sets and let $f \colon A \to B$ and $g \colon B \to C$ be functions.
a. Show that if $g \circ f$ is injective, then f is injective.
b. Show that if $g \circ f$ is surjective and g is injective, then f is surjective.

Problem 7: Let Q and P be defined as in Section 3.5 of the notes. Thus, Q is the set of equivalence classes of the set $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ under the equivalence relation $(a, b) \sim (c, d)$ if $ad = bc$, and P is the set of equivalence classes of the set $\mathbb{R}^2\backslash\{(0, 0)\}$ under the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if there exists a real number $\lambda \neq 0$ with $(x_1, y_1) = (\lambda x_2, \lambda y_2)$. Determine which of the following functions on equivalence classes are well-defined. In each case, either give a proof or a specific counterexample.
a. $f \colon Q \to \mathbb{Z}$ defined by $f(\overline{(a, b)}) = a - b$.
b. $f \colon Q \to Q$ defined by $f(\overline{(a, b)}) = \overline{(a^2 + 3ab + b^2, 5b^2)}$.
c. $f \colon P \to \mathbb{R}$ defined by

$$f(\overline{(x, y)}) = \frac{2xy^3 + 5xy}{x^4 + y^4}$$

d. $f \colon P \to P$ defined by $f(\overline{(x, y)}) = \overline{(x^3 + 5xy^2, y^3)}$.