Homework 11: Due Wednesday, May 10

Problem 1: Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$:
a. $x^4 - 5x^3 + 3x - 2$.
b. $x^4 - 2x^3 + 2x^2 + x + 4$.

Problem 2:
a. Find, with proof, all irreducible polynomials in $\mathbb{Z}/2\mathbb{Z}[x]$ of degree 2 or 3.
b. Show that $x^5 + x^2 + \overline{1} \in \mathbb{Z}/2\mathbb{Z}[x]$ is irreducible.

Problem 3: Let R be an integral domain. Suppose that $p, q \in R$ are associates.
a. Show that if p is irreducible, then q is irreducible.
b. Show that if p is prime, then q is prime.

Problem 4: Show that if R is a UFD, then every irreducible element of R is prime.
Aside: Theorem 11.5.12 says that if R is an integral domain where $\|$ is well-founded, and every irreducible is prime, then R is a UFD. This problem is a partial converse.

Problem 5: Suppose that R is a PID, i.e. an integral domain in which every ideal is principal. Let $a, b \in R$. Show that there exists a least common multiple of a and b. That is, show that there exists $c \in R$ with the following properties:

- $a \mid c$ and $b \mid c$.
- Whenever $d \in R$ satisfies both $a \mid d$ and $b \mid d$, it follows that $c \mid d$.

Hint: Think about the set of common multiples of a and b and how you can describe it as an ideal.

Problem 6: Working in the ring $\mathbb{Z}[x]$, let I be the ideal

$$I = \langle 2, x \rangle = \{p(x) \cdot 2 + q(x) \cdot x : p(x), q(x) \in \mathbb{Z}[x]\}.$$

Show that I is not a principal ideal in $\mathbb{Z}[x]$, and hence $\mathbb{Z}[x]$ is not a PID.