Homework 10: Due Wednesday, May 3

Problem 1: Let R be a ring. An element $a \in R$ is called *nilpotent* if there exists $n \in \mathbb{N}^+$ with $a^n = 0$.
a. Show that every nonzero nilpotent element is a zero divisor.
b. Show that if a is both nilpotent and idempotent, then $a = 0$.
c. Find (with explanation) all nilpotent elements in $\mathbb{Z}/36\mathbb{Z}$.

Problem 2: Let P be a prime ideal of a commutative ring R.
a. Let $a \in R$ and let $n \in \mathbb{N}^+$. Show that if $a^n \in P$, then $a \in P$.
b. Show that $a \in P$ for every nilpotent element $a \in R$.

Problem 3: Let $C[0,1]$ be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$. Define $+$ and $\cdot$ on $C[0,1]$ to be the usual (pointwise) addition and multiplication of functions. That is, we define

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (f \cdot g)(x) = f(x) \cdot g(x).$$

With these operations, $C[0,1]$ is a ring (the additive identity is the constant function 0, and the multiplicative identity is the constant function 1). Let

$$I = \{f \in C[0,1] : f(0) = 0 = f(1)\}.$$

a. Show that I is an ideal of $C[0,1]$.
b. Show that I is not a prime ideal of $C[0,1]$.

Problem 4: Suppose that R is a commutative ring with $|R| = 30$, and that I is an ideal of R with $|I| = 10$. Show that I is a maximal ideal of R.
Hint: An ideal is, in particular, an additive subgroup of the ring.

Problem 5: Show that the only ideals of $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.

Problem 6: Let $p \in \mathbb{N}^+$ be prime. Consider the polynomial $f(x) = x^p - x$ in $\mathbb{Z}/p\mathbb{Z}[x]$. How many roots does $f(x)$ have in $\mathbb{Z}/p\mathbb{Z}$? Explain.