Homework 1: Due Wednesday, February 1

Problem 1: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid 2b$ and $a \mid b + c$. Show that $a \mid 2c$.

Problem 2: Define a sequence recursively as follows. Let $a_0 = 6$, let $a_1 = 33$, and let $a_n = 7a_{n-1} - 2a_{n-2}$ for all $n \geq 2$. Use strong induction to show that $3 \mid a_n$ for all $n \in \mathbb{N}$. Be sure to state your inductive hypothesis carefully!

Problem 3: Show that $\mathsf{Div}(a) = \mathsf{Div}(-a)$ for all $a \in \mathbb{Z}$.

Problem 4: Use the Euclidean Algorithm to compute $\gcd(406, 182)$, and then use your computation to find $k, \ell \in \mathbb{Z}$ such that $406k + 182\ell = \gcd(406, 182)$.

Problem 5: Given $a \in \mathbb{Z}$, let $\mathsf{Mult}(a) = \{n \in \mathbb{Z} : a \mid n\}$ be the set of all multiples of a. Suppose now that $a, b \in \mathbb{Z}$ are both nonzero. Let

$$S = \mathsf{Mult}(a) \cap \mathsf{Mult}(b)$$

be the set of common multiples of a and b.
a. Explain why $S \cap \mathbb{N}^+ \neq \emptyset$, i.e. S contains a strictly positive element. For the rest of this problem, let m be the least element of $S \cap \mathbb{N}^+$, which exists by well-ordering.
b. Show that $a \mid m$ and $b \mid m$.
c. Suppose that $n \in \mathbb{Z}$ satisfies both $a \mid n$ and $b \mid n$. Show that $m \mid n$.
Hint for (c): Use Division with Remainder to fix $q, r \in \mathbb{Z}$ with $n = qm + r$ and $0 \leq r < m$. Argue that it must be the case that $r = 0$.
Note: Due to parts (b) and (c), the element m is called the *least common multiple* of a and b.

Problem 6: Find, with proof, all $n \in \mathbb{Z}$ such that $\gcd(n, n + 2) = 2$.

Problem 7: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid c$, that $b \mid c$, and that $\gcd(a, b) = 1$. Using only the material through Section 2.4 (so without using any properties of prime factorizations), show that $ab \mid c$.