Homework 9 : Due Monday, March 10

Problem 1: Determine, with proof, whether the following pairs of groups are isomorphic.
a. A_6 and S_5.
b. $\mathbb{Z}/84\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$.
c. $U(\mathbb{Z}/18\mathbb{Z})$ and $\mathbb{Z}/6\mathbb{Z}$.
d. S_4 and $\mathbb{Z}/6\mathbb{Z} \times U(\mathbb{Z}/5\mathbb{Z})$.
e. A_4 and D_6.
f. $U(\mathbb{Z}/5\mathbb{Z})$ and $U(\mathbb{Z}/10\mathbb{Z})$.
g. $S_3 \times \mathbb{Z}/2\mathbb{Z}$ and A_4.
h. $D_4/Z(D_4)$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Hint: Don't try to build explicit isomorphisms or rule out each possibility. Use the theory we have developed.

Problem 2: Consider the group $G = U(\mathbb{Z}/15\mathbb{Z})$. Find nontrivial cyclic subgroups H and K of G such that G is the internal direct product of H and K. Use this to find $m, n \in \mathbb{N}$ with $m, n \geq 2$ such that $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Problem 3: Let G be a group and let H be a normal subgroup of G.
a. Show that G/H is abelian if and only if $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.
b. Suppose $[G : H]$ is finite and let $m = [G : H]$. Show that $a^m \in H$ for all $a \in G$.

Problem 4:
a. Let G be a group and let H be a subgroup of G. Let $g \in G$. Show that the set

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

is a subgroup of G and that $|gHg^{-1}| = |H|$ (where $|A|$ means the number of elements in the set A).
b. Let G be a group. Suppose that $k \in \mathbb{N}^+$ is such that G has a unique subgroup of order k. If H is the unique subgroup of G of order k, show that H is a normal subgroup of G.