Homework 8 : Due Wednesday, March 5

Problem 1: For each of the following subgroups H of the given group G, determine if H is a normal subgroup of G.
a. $G = S_4$ and $H = \langle(1\ 2\ 3\ 4)\rangle = \{id, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$.
b. $G = D_4$ and $H = \langle rs\rangle = \{id, rs\}$.
c. $G = Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ and $H = \{1, -1\}$.

Problem 2: Suppose that H and K are both normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.

Problem 3: Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.
Note: We argued in class that $[\mathbb{Q} : \mathbb{Z}] = \infty$ (if $q, r \in \mathbb{Q}$ with $0 \leq q < r < 1$ then $q + \mathbb{Z} \neq r + \mathbb{Z}$ because $r - q \notin \mathbb{Z}$), so $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group.

Problem 4:
a. Suppose that G is a group with $|G| \neq 1$ and $|G|$ not prime (so either $|G|$ is composite and greater than 1, or $|G| = \infty$). Show that there exists a subgroup H of G with $H \neq \{e\}$ and $H \neq G$.
b. Show that the only abelian simple groups are the cyclic groups of prime order.

Problem 5: Let G and H be groups. Show that $G \times H \cong H \times G$.

Problem 6: Consider the group $G = \mathbb{R}\backslash\{-1\}$ with operation $a * b = a + b + ab$ from Homework 3. Let H be the group $\mathbb{R}\backslash\{0\}$ with operation equal to the usual multiplication. Show that $G \cong H$.