Homework 7 : Due Monday, February 24

Problem 1: Let $G = A_4$ and $H = \langle(1\ 2\ 3)\rangle = \{id, (1\ 2\ 3), (1\ 3\ 2)\}$. Compute the *left and right* cosets of H in G.

Problem 2: Let G be a finite group with $|G| = n$, and let $H = \{(a, a) : a \in G\}$.
a. Show that H is a subgroup of $G \times G$.
b. Compute $[G \times G : H]$.

Problem 3: Let H be a subgroup of G and let $a \in G$. Show that if $aH = Hb$ for some $b \in G$, then $aH = Ha$. In other words, if the left coset aH equals *some* right coset of H in G, then it must equal the right coset Ha.
Hint: Use the general theory of equivalence relations to simplify your life.

Problem 4: Suppose that H is a subgroup of a group G with $[G : H] = 2$. Suppose that $a, b \in G$ with both $a \notin H$ and $b \notin H$. Show that $ab \in H$.
Hint: Think about the four cosets eH, aH, bH, and abH.

Problem 5: Let G be a group and let H and K be subgroups of G. Let $a \in G$. Show that the two sets $aH \cap aK$ and $a(H \cap K)$ are equal. Thus, the left cosets of the subgroup $H \cap K$ are obtained by intersecting the corresponding left cosets of H and K individually.

Problem 6: Let G be a group. Suppose that H and K are finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e\}$.
Hint: Try to show that $|H \cap K| = 1$.

Problem 7: Suppose that G and H are finite groups. Show that if $|G|$ and $|H|$ are not relatively prime, then $G \times H$ is not cyclic (even if G and H are cyclic).
Note: This is the converse to Problem 3b on Homework 5.