Homework 6 : Due Wednesday, February 19

Problem 1: Let $n \geq 3$. Show that every element of A_n can be written as a product of 3-cycles (so the set of 3-cycles generates A_n).

Problem 2: Suppose that $\sigma \in A_n$ and $|\sigma| = 2$. Show that there exists $\tau \in S_n$ with $|\tau| = 4$ and $\tau^2 = \sigma$.

Problem 3: This problem gives another interpretation of D_n as a subgroup of $GL_2(\mathbb{R})$ by thinking of rotation and flips as linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.
a. Let $\alpha, \beta \in \mathbb{R}$. Show that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

b. Let $n \geq 3$. Let

$$R = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \qquad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $|R| = n$, $|S| = 2$, and $SR = R^{-1}S$.

Problem 4: Let $n \geq 3$. Working in D_n, determine $|r^k s^\ell|$ for each $k, \ell \in \mathbb{N}$ with $0 \leq k \leq n-1$ and $0 \leq \ell \leq 1$.

Problem 5: Let $n \geq 3$.
a. Show that if $a \in D_n$ and $a \in \langle r \rangle$, then $sa = a^{-1}s$.
b. Show that if $a \in D_n$ but $a \notin \langle r \rangle$, then $ra = ar^{-1}$.
c. Find $Z(D_n)$. You answer will depend on whether n is even or odd.

Problem 6: Compute the left cosets of the subgroup H of the given group G in each of the following cases (make sure you completely determine H first!).
a. $G = U(\mathbb{Z}/18\mathbb{Z})$ and $H = \langle \overline{17} \rangle$ (you computed the Cayley table of $U(\mathbb{Z}/18\mathbb{Z})$ in Homework 4).
b. $G = D_4$ and $H = \langle r^2 s \rangle$.
Hint: Save as much work as you can by using the general fact that you are working with equivalence classes of a certain equivalence relation, and you know that the equivalence classes partition G.