Homework 5 : Due Friday, February 14

Problem 1: Find the order of the following elements in the given direct product.
a. $((1\ 6\ 4)(3\ 7), (1\ 4\ 2\ 3)) \in S_9 \times S_4$
b. $(\overline{5}, \overline{7}, \overline{44}) \in \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/84\mathbb{Z}$
c. $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \overline{3}) \in GL_2(\mathbb{R}) \times U(\mathbb{Z}/13\mathbb{Z})$

Problem 2:
a. Show that if G is an abelian group and $a, b \in G$ have finite order, then $|ab| \leq \text{lcm}(|a|, |b|)$.
b. Give an example of an abelian group G and elements $a, b \in G$ with finite order such that $|ab| < \text{lcm}(|a|, |b|)$.

Problem 3: Suppose that G and H are groups.
a. Show that if $G \times H$ is cyclic, then both G and H are cyclic.
b. Suppose that G and H are both finite and cyclic. Show that if $|G|$ and $|H|$ are relatively prime, then $G \times H$ is cyclic.

Problem 4: Let $n \in \mathbb{N}$ with $n \geq 2$.
a. Show that $\{(1\ \ a) : 2 \leq a \leq n\}$ generates S_n.
b. Show that $\{(a\ \ a+1) : 1 \leq a \leq n-1\}$ generates S_n.
c. Show that $\{(1\ \ 2), (1\ \ 2\ \ 3\ \cdots\ n)\}$ generates S_n.
Hint: Don't reinvent the wheel every time. You already know you can get everything from the transpositions. Once you've done part a, you know you can get everything from that smaller set, etc.

Problem 5:
a. Suppose that G is an abelian group and that $a_1, a_2, \ldots, a_n \in G$. Show that

$$\langle a_1, a_2, \ldots, a_n \rangle = \{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} : k_1, k_2, \ldots, k_n \in \mathbb{Z}\}.$$

b. Consider the group $\mathbb{Z}$ (under addition). Let $a, b \in \mathbb{Z}$, and let $d = \gcd(a, b)$. Show that $\langle a, b \rangle = \langle d \rangle$.
c. Show that the group $\mathbb{Q}$ (under addition) is not finitely generated. In other words, show that there do not exist $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ such that $\langle q_1, q_2, \ldots, q_n \rangle = \mathbb{Q}$.
Hint for b and c: Use part a, together with the fact that in $\mathbb{Z}$ and $\mathbb{Q}$, the group theory notation a^k means ka and the operation is addition.