Homework 3 : Due Wednesday, February 5

Problem 1: Let Q and P be defined as in section 3.5 of the notes. Thus, Q is the set of equivalence classes of the set $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ under the equivalence relation $(a, b) \sim (c, d)$ if $ad = bc$, and P is the set of equivalence classes of the set $\mathbb{R}^2 \backslash \{(0, 0)\}$ under the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if there exists a real number $\lambda \neq 0$ with $(x_2, y_2) = (\lambda x_1, \lambda y_1)$. Determine which of the following functions on equivalence classes are well-defined. In each case, either give a proof or a specific counterexample.
a. $f\colon Q \to \mathbb{Z}$ defined by $f(\overline{(a, b)}) = a - b$.
b. $f\colon Q \to Q$ defined by $f(\overline{(a, b)}) = \overline{(a^2 + 3ab + b^2, 5b^2)}$.
c. $f\colon P \to \mathbb{R}$ defined by

$$f(\overline{(x, y)}) = \frac{2xy^3 + 5xy}{x^4 + y^4}$$

d. $f\colon P \to P$ defined by $f(\overline{(x, y)}) = \overline{(x^3 + 5xy^2, y^3)}$.

Problem 2: Let $\times$ be the cross product on $\mathbb{R}^3$.
a. Is $\times$ an associative operation on $\mathbb{R}^3$? Either prove or give an explicit counterexample.
b. Does $\times$ have an identity on $\mathbb{R}^3$? Prove your answer.

Problem 3: Consider the set $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ of nonnegative reals. Let $*$ be the binary operation on $\mathbb{R}^{\geq 0}$ given by exponentiation, i.e. $a * b = a^b$.
a. Is $*$ an associative operation on $\mathbb{R}^{\geq 0}$? Either prove or give an explicit counterexample.
b. Does $*$ have an identity on $\mathbb{R}^{\geq 0}$? Either prove or give an explicit counterexample.

Problem 4: Define a binary operation $*$ on $\mathbb{R}$ by letting $a * b = a + b + ab$.
a. Show that $*$ is commutative, i.e. that $a * b = b * a$ for all $a, b \in \mathbb{R}$.
b. Show that $*$ is associative, i.e. that $(a * b) * c = (a * b) * c$ for all $a, b, c \in \mathbb{R}$.
c. Show that $\mathbb{R}$ with operation $*$ has an identity element.
d. Show that the set of invertible elements of $*$ equals $\mathbb{R} \backslash \{-1\} = \{x \in \mathbb{R} : x \neq -1\}$.
Note: Using Corollary 4.3.5, it follows that $\mathbb{R} \backslash \{-1\}$ under $*$ and with the identity element from part c is an abelian group.

Problem 5: Let S be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

where $a \in \mathbb{R}$ and $a \neq 0$.
a. Show that if $A, B \in S$, then $AB \in S$. Thus, matrix multiplication is a binary operation on S.
b. Show that S with matrix multiplication has an identity element.
c. Notice that every matrix in S has determinant 0, so every matrix in S fails to be invertible in the linear algebra sense. Nevertheless, show that S is group under matrix multiplication with the identity from part b.

Problem 6: Let G be a group. Suppose that $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ for all $a, b \in G$. Show that G is abelian.