Homework 2 : Due Friday, January 31

Note: For the first four problems, use only the material through Section 2.4, so do not use any properties of prime factorizations.

Problem 1: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid c$, that $b \mid c$, and that $\gcd(a, b) = 1$. Show that $ab \mid c$.

Problem 2: Let $a, b, c \in \mathbb{Z}$. Show that the following are equivalent (in other words, prove that 1 implies 2 and also that 2 implies 1):

1. $\gcd(ab, c) = 1$
2. $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$

Problem 3: Find, with proof, all $n \in \mathbb{Z}$ such that $\gcd(n, n + 2) = 2$.

Problem 4: Let $a, b \in \mathbb{N}^+$ and let $d = \gcd(a, b)$. Since d is a common divisor of a and b, we may fix $k, \ell \in \mathbb{N}$ with $a = kd$ and $b = \ell d$. Let $m = k\ell d$.
a. Show that $a \mid m$, $b \mid m$, and $dm = ab$.
b. Show that $\gcd(k, \ell) = 1$.
c. Suppose that $n \in \mathbb{Z}$ is such that $a \mid n$ and $b \mid n$. Show that $m \mid n$.
Because of parts a and c above, the number m is called the *least common multiple* of a and b and is written as $\operatorname{lcm}(a, b)$. Since $dm = ab$ from part a, it follows that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

Problem 5: Let $S = \{2n : n \in \mathbb{Z}\}$ be the set of even integers. Notice that the sum and product of two elements of S is still an element of S. Call an element of $a \in S$ *irreducible* if $a > 0$ and there is no way to write $a = bc$ with $b, c \in S$. Notice that 6 is irreducible in S even though it is not prime in $\mathbb{Z}$ (although $6 = 2 \cdot 3$, we have that $3 \notin S$).
a. Give a characterization of the irreducible elements of S.
b. Show that the analogue of Fundamental Theorem of Arithmetic fails in S by finding a positive element of S which does *not* factor uniquely (up to order) into irreducibles.

Problem 6: Let $A = \mathbb{N}$ and $a \sim b$ if and only if there exists $n \in \mathbb{Z}$ with $a = 2^n b$.
a. Show that $\sim$ is an equivalence relation on A.
b. Characterize which elements of $\mathbb{N}$ are the smallest elements of their equivalence class.