Homework 15: Due Friday, May 2

Problem 1: Recall that a Boolean ring is a ring R for which $a^2 = a$ for all $a \in R$. In Homework 12, we proved every Boolean ring is commutative.
a. Show that if R is both a Boolean ring and an integral domain, then $R \cong \mathbb{Z}/2\mathbb{Z}$.
b. Show that if R is a Boolean ring and I is an ideal of R, then R/I is a Boolean ring.
c. Show that every prime ideal in a Boolean ring is a maximal ideal.

Problem 2: Let R be an integral domain.
a. Show that every associate of an irreducible element of R is irreducible.
b. Show that every associate of a prime element of R is prime.

Problem 3:
a. Find, with proof, all irreducible polynomials in $\mathbb{Z}/2\mathbb{Z}[x]$ of degree 2 or 3.
b. Show that $x^5 + x^2 + \overline{1} \in \mathbb{Z}/2\mathbb{Z}[x]$ is irreducible.

Problem 4: Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$.
a. $x^4 - 5x^3 + 3x - 2$
b. $x^4 - 2x^3 + 2x^2 + x + 4$

Problem 5: Let R be an integral domain and let $a, c, d \in R$. Show that if c and d are associates in R, then $ord_c(a) = ord_d(a)$.

Problem 6: Show that if R is a UFD, then every irreducible element of R is prime.