Homework 14 : Due Friday, April 18

Problem 1: Let R be a ring and let I and J be ideals of R. Define the following set:

$$I + J = \{c + d : c \in I, d \in J\}$$

Prove that $I + J$ is an ideal of R (it is the smallest ideal of R containing both I and J).

Problem 2: Consider the subring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of $\mathbb{R}$.
a. Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$.
b. Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

Problem 3: Let $p \in \mathbb{N}^+$ be prime. Consider the polynomial $f(x) = x^p - x$ in $\mathbb{Z}/p\mathbb{Z}[x]$. How many roots does $f(x)$ have in $\mathbb{Z}/p\mathbb{Z}$? Explain.

Problem 4: Working in the ring $\mathbb{Z}[x]$, let I be the ideal

$$I = \langle 2, x \rangle = \{p(x) \cdot 2 + q(x) \cdot x : p(x), q(x) \in \mathbb{Z}[x]\}$$

Show that I is not a principal ideal in $\mathbb{Z}[x]$.

Problem 5: Recall from Homework 12 that $a \in R$ is *nilpotent* if there exists $n \in \mathbb{N}^+$ with $a^n = 0$. Let R be a commutative ring and let P be a prime ideal of R. Show that $a \in P$ for every nilpotent element $a \in R$.

Problem 6: Let $C[0,1]$ be set of all continuous functions $f \colon [0,1] \to \mathbb{R}$. Define $+$ and $\cdot$ on $C[0,1]$ to be the usual addition and multiplication of functions. That is, we define

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (f \cdot g)(x) = f(x) \cdot g(x)$$

With these operations, $C[0,1]$ is a ring (the additive identity is the constant function 0, and the multiplicative identity is the constant function 1). Let

$$I = \{f \in C[0,1] : f(0) = 0 = f(1)\}$$

a. Show that I is an ideal of $C[0,1]$.
b. Show that I is not a prime ideal of $C[0,1]$.