Homework 13 : Due Monday, April 14

Problem 1: For each of the following fields F, and given $f(x), g(x) \in F[x]$, calculate the unique $q(x), r(x) \in F[x]$ with $f(x) = q(x)g(x) + r(x)$ and either $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.
a. $F = \mathbb{Z}/2\mathbb{Z}$: $f(x) = x^5 + x^3 + x^2 + \bar{1}$ and $g(x) = x^2 + x$.
b. $F = \mathbb{Z}/5\mathbb{Z}$: $f(x) = x^3 + \bar{3}x^2 + \bar{2}$ and $g(x) = \bar{4}x^2 + \bar{1}$

Problem 2: Find a nonconstant polynomial in $\mathbb{Z}/4\mathbb{Z}[x]$ which is a unit. Moreover, show that for every $n \in \mathbb{N}^+$, there exists a polynomial in $\mathbb{Z}/4\mathbb{Z}[x]$ of degree n which is a unit.

Problem 3: Consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ as a direct product (so addition and multiplication are componentwise). Determine, with explanation, which of the following subsets are ideals of R.
a. $\{(a, 0) : a \in \mathbb{Z}\}$
b. $\{(a, a) : a \in \mathbb{Z}\}$
c. $\{(2a, 3b) : a, b \in \mathbb{Z}\}$

Problem 4: Define $\varphi \colon \mathbb{C} \to M_2(\mathbb{R})$ by letting

$$\varphi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Show that φ is an injective ring homomorphism (so $\mathbb{C}$ is isomorphic to the subring range(φ) of $M_2(\mathbb{R})$).

Problem 5: Suppose that R and S are rings and that $\varphi \colon R \to S$ is a function such that

- $\varphi(r + s) = \varphi(r) + \varphi(s)$ for all $r, s \in R$.

- $\varphi(rs) = \varphi(r) \cdot \varphi(s)$ for all $r, s \in R$.

Thus, in contrast to the definition of a ring homomorphism, we are *not* assuming that $\varphi(1_R) = 1_S$.
a. Show that $\varphi(1_R)$ is an idempotent of S.
b. Show that if φ is surjective, then $\varphi(1_R) = 1_S$.
c. Suppose that S is an integral domain and that φ is not the zero function (i.e. there exists $r \in R$ with $\varphi(r) \neq 0_S$). Show that $\varphi(1_R) = 1_S$.

Problem 6: Consider $\mathbb{R}$ and $\mathbb{C}$ as rings. Show that $\mathbb{R} \not\cong \mathbb{C}$.

Problem 7: Show that the only ideals of $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.
Hint: Suppose that $I \neq \{0\}$. First show that I contains an invertible matrix.