

Homework 10 : Due Friday, March 14

Problem 1: Let $G = (\mathbb{R}, +)$ and let $H = (\mathbb{R} \setminus \{0\}, \cdot)$. Show that $G \not\cong H$.

Problem 2: Let G and H be groups and let $\varphi: G \rightarrow H$ and $\psi: G \rightarrow H$ be homomorphisms. Show that $\{g \in G : \varphi(g) = \psi(g)\}$ is a subgroup of G .

Note: It follows that if $G = \langle c \rangle$, and $\varphi(c) = \psi(c)$, then $\varphi = \psi$ (because the smallest subgroup of G containing c is all of G). Similarly, if $A \subseteq G$ is such that $G = \langle A \rangle$, and $\varphi(a) = \psi(a)$ for all $a \in A$, then $\varphi = \psi$.

Problem 3: Given a group G , consider the group $G \times G$ and the subset $D = \{(a, a) : a \in G\}$. It is straightforward to check that D is a subgroup of $G \times G$ and that $D \cong G$.

a. Show that if $G = S_3$, then D is not a normal subgroup of $G \times G$.

b. Suppose that G is abelian. Find a surjective homomorphism $\varphi: G \times G \rightarrow G$ with $\ker(\varphi) = D$ and use it to conclude that $(G \times G)/D \cong G$.

Problem 4: Let $G = \mathbb{Z}/24\mathbb{Z}$, let $H = \langle \bar{4} \rangle$ and let $N = \langle \bar{6} \rangle$. The Second Isomorphism Theorem says that

$$\frac{H}{H \cap N} \cong \frac{H + N}{N}$$

(note that we wrote $H + N$ rather than HN because the group operation is addition).

a. Explicitly calculate $H \cap N$ and $H + N$.

b. Explicitly list the cosets in the groups $H/(H \cap N)$ and $(H + N)/N$.

c. Follow the proof of the Second Isomorphism Theorem to explicitly write down the isomorphism from $H/(H \cap N)$ to $(H + N)/N$ using your descriptions of the elements in part b.

Problem 5: Let $G = \mathbb{R}$ (under addition) and let $X = \mathbb{R}^2$. Define a function from $G \times X$ to X by $a * (x, y) = (x + ay, y)$.

a. Show that $*$ is an action of G on X .

b. Describe the orbits of the action geometrically. Be careful!

c. Describe the stabilizers of each element of X .

Problem 6: Let $G = S_3$ and let

$$X = \{1, 2, 3\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Define a function from $G \times X$ to X by $\sigma * (x, y) = (\sigma(x), \sigma(y))$.

a. Show that $*$ is an action of G on X .

b. Find the orbits and stabilizers of each element of X .