Homework 7: Due Tuesday, September 29

Exercises

Exercise 1: Let $G = (\mathbb{R}, +)$ and let $H = (\mathbb{R}\backslash\{0\}, \cdot)$. Show that $G \ncong H$.

Exercise 2: Let G be a cyclic group. Show that if H is a subgroup of G, then H is a normal subgroup of G and G/H is cyclic.

Exercise 3: Either prove or give a counterexample: Let G be a group and let H be a normal subgroup of G. If both H and G/H are abelian, then G is abelian.

Problems

Problem 1:
a. Suppose that G is a group with $|G| \neq 1$ and $|G|$ not prime (so either $|G|$ is composite and greater than 1, or $|G| = \infty$). Show that there exists a subgroup H of G with $H \neq \{e\}$ and $H \neq G$.
b. Suppose that G is an *abelian* group. Show that G is simple if and only if $|G|$ is prime.

Problem 2: Let G and H be groups and let $\varphi\colon G \to H$ and $\psi\colon G \to H$ be homomorphisms. Show that $\{g \in G : \varphi(g) = \psi(g)\}$ is a subgroup of G.
Note: It follows that if $G = \langle c \rangle$, and $\varphi(c) = \psi(c)$, then $\varphi = \psi$ (because the smallest subgroup of G containing c is the entire group G). Similarly, if $A \subseteq G$ is such that $G = \langle A \rangle$, and $\varphi(a) = \psi(a)$ for all $a \in A$, then $\varphi = \psi$.

Problem 3: Given a group G, consider the group $G \times G$ and the subset $D = \{(a, a) : a \in G\}$. On Homework 5, you showed that D is a subgroup of $G \times G$, and it is straightforward to check that $D \cong G$.
a. Show that if $G = S_3$, then D is not a normal subgroup of $G \times G$.
b. Suppose that G is abelian. Find a surjective homomorphism $\varphi\colon G \times G \to G$ with $\ker(\varphi) = D$ and use it to conclude that $(G \times G)/D \cong G$.

Problem 4: An *automorphism* of a group G is an isomorphism $\varphi\colon G \to G$. Let G be a group, and fix $g \in G$. Define a function $\varphi_g\colon G \to G$ by letting $\varphi_g(a) = gag^{-1}$. Show that φ_g is an automorphism of G.

Problem 5: Let $G = \mathbb{R}$ (under addition) and let $X = \mathbb{R}^2$. Define a function from $G \times X$ to X by $a * (x, y) = (x + ay, y)$.
a. Show that $*$ is a action of G on X.
b. Describe the orbits of the action geometrically. Be careful!
c. Describe the stabilizers of each element of X.

Problem 6: Let $G = S_3$ and let

$$X = \{1, 2, 3\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Define a function from $G \times X$ to X by $\sigma * (x, y) = (\sigma(x), \sigma(y))$.
a. Show that $*$ is an action of G on X.
b. Find the orbits and stabilizers of each element of X.