Homework 6: Due Friday, September 25

Exercises

Exercise 1: Let G be a group and let H be a normal subgroup of G. Show that G/H is abelian if and only if $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.

Exercise 2: Let G be a group and let H be a subgroup of G. Given $g \in G$, define the set

$$gHg^{-1} = \{ghg^{-1} : h \in H\}.$$

a. Show that gHg^{-1} is a subgroup of G.
b. Show that there is a bijection between H and gHg^{-1} (so, assuming that H is finite, the sets H and gHg^{-1} have the same number of elements).
c. Let G be a group. Suppose that $k \in \mathbb{N}^+$ is such that G has a unique subgroup of order k. If H is the unique subgroup of G of order k, show that H is a normal subgroup of G.

Problems

Problem 1: Determine, with proof, whether the following pairs of groups are isomorphic.
a. A_6 and S_5.
b. $\mathbb{Z}/84\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$.
c. $U(\mathbb{Z}/18\mathbb{Z})$ and $\mathbb{Z}/6\mathbb{Z}$.
d. S_4 and $\mathbb{Z}/6\mathbb{Z} \times U(\mathbb{Z}/5\mathbb{Z})$.
e. A_4 and D_6.
f. $U(\mathbb{Z}/5\mathbb{Z})$ and $U(\mathbb{Z}/10\mathbb{Z})$.
g. $S_3 \times \mathbb{Z}/2\mathbb{Z}$ and A_4.
h. $D_4/Z(D_4)$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 2: Let G and H be groups. Show that $G \times H \cong H \times G$.

Problem 3: Consider the group $G = \mathbb{R}\backslash\{-1\}$ with operation $a * b = a + b + ab$ from Homework 3. Let H be the group $\mathbb{R}\backslash\{0\}$ with operation equal to the usual multiplication. Show that $G \cong H$.

Problem 4: Consider the group $G = U(\mathbb{Z}/15\mathbb{Z})$. Find nontrivial cyclic subgroups H and K of G such that G is the internal direct product of H and K. Use this to find $m, n \in \mathbb{N}$ with $m, n \geq 2$ such that $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Problem 5: Let G be a group and let H be a normal subgroup of G. Suppose $[G : H]$ is finite and let $m = [G : H]$. Show that $a^m \in H$ for all $a \in G$.