Homework 5: Due Tuesday, September 22

Exercises

Exercise 1: For each of the following subgroups H of the given group G, determine if H is a normal subgroup of G.
a. $G = S_4$ and $H = \langle (1\ 2\ 3\ 4) \rangle = \{id, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$.
b. $G = D_4$ and $H = \langle rs \rangle = \{e, rs\}$.
c. $G = A_4$ and $H = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. (First check that H is indeed a subgroup of G).
Suggestion: Normal subgroups have many equivalent characterizations. In each part, pick one of these which will make your life easy.

Exercise 2: Let H be a subgroup of G and let $a \in G$. Show that if $aH = Hb$ for some $b \in G$, then $aH = Ha$. In other words, if the left coset aH equals *some* right coset of H in G, then it must equal the right coset Ha.
Hint: Use the general theory of equivalence relations to simplify your life.

Exercise 3: Suppose that G and H are finite groups. Show that if $|G|$ and $|H|$ are not relatively prime, then $G \times H$ is not cyclic (regardless of whether G and H are cyclic).
Note: This is the converse to Problem 6c on Homework 4.

Problems

Problem 1: Let G be a group and let $H = \{(a, a) : a \in G\}$.
a. Show that H is a subgroup of $G \times G$.
b. Assuming G is a finite group with $|G| = n$, compute $[G \times G : H]$.

Problem 2: Let G be a group. Suppose that H and K are finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e\}$.
Hint: Make use of Lagrange's Theorem.

Problem 3: Determine both the left cosets and the right cosets of the subgroup H of the given group G in each of the following cases (make sure you completely determine H first!).
a. $G = D_4$ and $H = \langle r^2 s \rangle$.
b. $G = A_4$ and $H = \langle (1\ 2\ 3) \rangle$.
Hint: Save as much work as you can by using the general fact that you are working with equivalence classes of a certain equivalence relation, and you know that the equivalence classes partition G.

Problem 4: Suppose that H and K are both normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.

Problem 5: Suppose that H is a subgroup of a group G with $[G : H] = 2$. Suppose that $a, b \in G$ with both $a \notin H$ and $b \notin H$. Show that $ab \in H$.
Hint: Think about the four cosets eH, aH, bH, and abH.

Problem 6: Consider the group $(\mathbb{Q}, +)$. Notice that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and in fact it is a normal subgroup of $\mathbb{Q}$ because $\mathbb{Q}$ is abelian. Thus, we can form the quotient $\mathbb{Q}/\mathbb{Z}$. In class, we mentioned that for every $q \in \mathbb{Q}$, there exists $r \in \mathbb{Q}$ with $0 \le r < 1$ such that $q + \mathbb{Z} = r + \mathbb{Z}$. For example, we have $\frac{5}{2} + \mathbb{Z} = \frac{1}{2} + \mathbb{Z}$ and $-\frac{1}{7} + \mathbb{Z} = \frac{6}{7} + \mathbb{Z}$. In other words, we have

$$\mathbb{Q}/\mathbb{Z} = \{r + \mathbb{Z} : r \in \mathbb{Q} \cap [0, 1)\}.$$

a. Show that if $r_1, r_2 \in \mathbb{Q}$ with $0 \leq r_1 < r_2 < 1$ then $r_1 + \mathbb{Z} \neq r_2 + \mathbb{Z}$.
b. Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.
Note: Thus, $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group in which every element has finite order.