Homework 11: Due Friday, October 16

Exercises

Exercise 1: Consider the subring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of $\mathbb{R}$.
a. Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$.
b. Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

Exercise 2: Determine, with proof, the number of elements in the quotient ring $\mathbb{Z}[i]/\langle 3\rangle$.

Exercise 3: Suppose that R is a PID, i.e. an integral domain in which every ideal is principal. Let $a, b \in R$. Show that there exists a least common multiple of a and b. That is, show that there exists $c \in R$ with the following properties:

- $a \mid c$ and $b \mid c$.
- Whenever $d \in R$ satisfies both $a \mid d$ and $b \mid d$, it follows that $c \mid d$.

Hint: Think about the set of common multiples of a and b and how you can describe it as an ideal.

Problems

Problem 1: Suppose that R is a commutative ring with $|R| = 30$, and that I is an ideal of R with $|I| = 10$. Show that I is a maximal ideal of R.
Hint: An ideal is, in particular, an additive subgroup of the ring. Some group theory might be useful here.

Problem 2: Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$.
a. $x^4 - 5x^3 + 3x - 2$.
b. $x^4 - 2x^3 + 2x^2 + x + 4$.

Problem 3:
a. Find, with proof, all irreducible polynomials in $\mathbb{Z}/2\mathbb{Z}[x]$ of degree 2 or 3.
b. Show that $x^5 + x^2 + \overline{1} \in \mathbb{Z}/2\mathbb{Z}[x]$ is irreducible.

Problem 4: Let R be an integral domain. Suppose that $p, q \in R$ are associates.
a. Show that if p is irreducible, then q is irreducible.
b. Show that if p is prime, then q is prime.

Problem 5: Show that if R is a UFD, then every irreducible element of R is prime.
Aside: Theorem 11.5.12 says that if R is an integral domain where $\|$ is well-founded, and every irreducible is prime, then R is a UFD. This problem is a partial converse.