Homework 1: Due Friday, September 4

Exercises

Exercise 1: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid b$ and $a \nmid c$. Show that $a \nmid (b + c)$.

Exercise 2: Use induction to show that $6 \mid (2n^3 + 3n^2 + n)$ for all $n \in \mathbb{N}$.

Exercise 3: Show that $Div(a) = Div(-a)$ for all $a \in \mathbb{Z}$.

Exercise 4: Use the Euclidean Algorithm to find the greatest common divisor of the following pairs of numbers a and b. Furthermore, once you find the greatest common divisor m, find $k, \ell \in \mathbb{Z}$ such that $ka + \ell b = m$.

- $a = 234$ and $b = 165$.
- $a = 562$ and $b = 471$.

Problems

Problem 1: Define a sequence recursively as follows. Let $a_0 = 6$, let $a_1 = 33$, and let $a_n = 7a_{n-1} - 2a_{n-2}$ for all $n \geq 2$. Use strong induction to show that $3 \mid a_n$ for all $n \in \mathbb{N}$. Be sure to state your inductive hypothesis carefully!

Problem 2: Let $a \in \mathbb{Z}$ with $5 \nmid a$. Show that the remainder when dividing a^2 by 5 is either 1 or 4, i.e. that either there exists $k \in \mathbb{Z}$ with $a^2 = 5k + 1$ or there exists $k \in \mathbb{Z}$ with $a^2 = 5k + 4$.
Hint: Start by performing division with remainder on a.

Problem 3: Let $n \in \mathbb{Z}$. Show that if $a \in \mathbb{Z}$ satisfies both $a \mid 3n^2 - 6n + 5$ and $a \mid 2n^2 - 4n + 1$, then $a \mid 7$.

Problem 4: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid c$, that $b \mid c$, and that $\gcd(a, b) = 1$. Using only the material through Section 2.4 (so without using any properties of prime factorizations), show that $ab \mid c$.

Problem 5: Show that $\{n \in \mathbb{Z} : \gcd(5n + 4, 10n + 6) = 1\} = \{n \in \mathbb{Z} : n \text{ is odd}\}$.