Homework 9: Due Friday, November 8

Problem 1: Compute, with explanation, the conjugacy classes and the Class Equation for D_5.
Note: You can really cut down on computations using the ideas from class. It is even possible to compute the Class Equation first and use it to do very few computations.

Problem 2: Determine which finite groups have exactly two conjugacy classes.

Problem 3: Suppose that G is a nonabelian group with $|G| = 125$. Show that $|Z(G)| = 5$ and that $G/Z(G) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 4: Let $n, p \in \mathbb{N}^+$ and assume that p is prime. Let

$$X = \{1, 2, \dots, n\}^p = \{(a_1, a_2, \dots, a_{p-1}, a_p) : 1 \leq a_i \leq n \text{ for all } i\}$$

be the set of all p-tuples such that each coordinate is an integer between 1 and n. Notice that S_p acts on X via

$$\sigma * (a_1, a_2, \dots, a_{p-1}, a_p) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(p-1)}, a_{\sigma(p)}).$$

Let $H = \langle (1\ 2\ \dots\ p) \rangle \subseteq S_p$, so $|H| = p$. Since H is a subgroup of S_p, we know that H acts on X via the above action as well. For example,

$$(1\ 2\ \dots\ p) * (a_1, a_2, \dots, a_{p-1}, a_p) = (a_2, a_3, \dots, a_p, a_1),$$

so $(1\ 2\ \dots\ p)$ cyclically shifts an element in X to the left by 1. Similarly, $(1\ 2\ \dots\ p)^2$ cyclically shifts to the left by 2, etc.
a. Show that every orbit of the action of H on X has size either 1 or p.
b. Show that there are exactly n orbits of size 1.
c. Show that $p \mid (n^p - n)$.
Note: This gives another proof of Fermat's Little Theorem.

Problem 5: This problem provides another proof of Cauchy's Theorem (so don't use Cauchy's Theorem in this problem!). Let G be a group and suppose that p is a prime which divides $|G|$. Let

$$X = \{(a_1, a_2, \dots, a_{p-1}, a_p) \in G^p : a_1 a_2 \cdots a_{p-1} a_p = e\},$$

i.e. X consists of all p-tuples of elements of G such that when you multiply them in the given order, you obtain the identity. For example, if $G = S_3$ and $p = 3$, then $((1\ 2), id, (1\ 2)) \in X$ and $((1\ 2), (1\ 3), (1\ 2\ 3)) \in X$, but $(id, (1\ 2), (1\ 2\ 3)) \notin X$.
a. Show that $|X| = |G|^{p-1}$.
b. Show that if $(a_1, a_2, \dots, a_{p-1}, a_p) \in X$, then $(a_2, a_3, \dots, a_p, a_1) \in X$.

Let $H = \langle (1\ 2\ \dots\ p) \rangle \subseteq S_p$, so $|H| = p$. Part (b) says that any cyclic shift of an element of X is also in X, so H acts on X as in Problem 4.

c. Notice that $|\mathcal{O}_{(e,e,\dots,e,e)}| = 1$. Show that there exists at least one other orbit of size 1.
d. Conclude that G has an element of order p.

Problem 6: Let R be a ring. An element $e \in R$ is called an *idempotent* if $e^2 = e$. Notice that 0 and 1 are idempotents in every ring R.
a. Show that if $e \in R$ is both a unit and an idempotent, then $e = 1$.

b. Show that if R is an integral domain, then 0 and 1 are the only idempotents of R.
c. Find all idempotents in $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/18\mathbb{Z}$.