Homework 6: Due Friday, October 11

Problem 1: For each of the following subgroups H of the given group G, determine if H is a normal subgroup of G.
a. $G = S_4$ and $H = \langle(1\ 2\ 3\ 4)\rangle = \{id, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$.
b. $G = D_4$ and $H = \langle rs\rangle = \{id, rs\}$.
c. $G = A_4$ and $H = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. (First check that H is indeed a subgroup of G).
Suggestion: Normal subgroups have many equivalent characterizations. In each part, pick one of these which will make your life easy.

Problem 2: Suppose that H and K are both normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.

Problem 3: Consider the group $(\mathbb{Q}, +)$. Notice that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and in fact it is a normal subgroup of $\mathbb{Q}$ because $\mathbb{Q}$ is abelian. Thus, we can form the quotient $\mathbb{Q}/\mathbb{Z}$. In class, we mentioned that for every $q \in \mathbb{Q}$, there exists $r \in \mathbb{Q}$ with $0 \leq r < 1$ such that $q + \mathbb{Z} = r + \mathbb{Z}$. For example, we have $\frac{5}{2} + \mathbb{Z} = \frac{1}{2} + \mathbb{Z}$ and $-\frac{1}{7} + \mathbb{Z} = \frac{6}{7} + \mathbb{Z}$. In other words, we have

$$\mathbb{Q}/\mathbb{Z} = \{r + \mathbb{Z} : r \in \mathbb{Q} \cap [0, 1)\}.$$

a. Show that if $r_1, r_2 \in \mathbb{Q}$ with $0 \leq r_1 < r_2 < 1$ then $r_1 + \mathbb{Z} \neq r_2 + \mathbb{Z}$.
b. Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.
Note: Thus, $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group in which every element has finite order.

Problem 4:
a. Suppose that G is a group with $|G| \neq 1$ and $|G|$ not prime (so either $|G|$ is composite and greater than 1, or $|G| = \infty$). Show that there exists a subgroup H of G with $H \neq \{e\}$ and $H \neq G$.
b. Suppose that G is an *abelian* group. Show that G is simple if and only if $|G|$ is prime.