Homework 5: Due Friday, October 4

Problem 1: Let $n \geq 3$. Working in D_n, determine $|r^k s^\ell|$ for each $k, \ell \in \mathbb{N}$ with $0 \leq k \leq n-1$ and $0 \leq \ell \leq 1$.

Problem 2: Let $n \geq 3$.
a. Show that if $a \in D_n$ and $a \in \langle r \rangle$, then $sa = a^{-1}s$.
b. Show that if $a \in D_n$ but $a \notin \langle r \rangle$, then $ra = ar^{-1}$.
c. Find $Z(D_n)$. Your answer will depend on whether n is even or odd.

Problem 3: Determine both the left cosets and the right cosets of the subgroup H of the given group G in each of the following cases (make sure you completely determine H first!).
a. $G = D_4$ and $H = \langle r^2 s \rangle$.
b. $G = A_4$ and $H = \langle (1\ 2\ 3) \rangle$.
Hint: Save as much work as you can by using the general fact that you are working with equivalence classes of a certain equivalence relation, and you know that the equivalence classes partition G.

Problem 4: Let G be a finite group with $|G| = n$, and let $H = \{(a, a) : a \in G\}$.
a. Show that H is a subgroup of $G \times G$.
b. Compute $[G \times G : H]$.

Problem 5: Let H be a subgroup of G and let $a \in G$. Show that if $aH = Hb$ for some $b \in G$, then $aH = Ha$. In other words, if the left coset aH equals *some* right coset of H in G, then it must equal the right coset Ha.
Hint: Use the general theory of equivalence relations to simplify your life.

Problem 6: Suppose that H is a subgroup of a group G with $[G : H] = 2$. Suppose that $a, b \in G$ with both $a \notin H$ and $b \notin H$. Show that $ab \in H$.
Hint: Think about the four cosets eH, aH, bH, and abH.

Problem 7: Let G be a group. Suppose that H and K are finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e\}$.
Hint: Make use of Lagrange's Theorem.

Problem 8: Suppose that G and H are finite groups. Show that if $|G|$ and $|H|$ are not relatively prime, then $G \times H$ is not cyclic (regardless of whether G and H are cyclic).
Note: This is the converse to Problem 6c on Homework 4.