Homework 3: Due Friday, September 20

Problem 1: Let $a, b \in \mathbb{N}^+$ and let $d = \gcd(a, b)$. Since d is a common divisor of a and b, we may fix $k, \ell \in \mathbb{N}$ with $a = kd$ and $b = \ell d$. Let $m = k\ell d$.
a. Show that $a \mid m$, $b \mid m$, and $dm = ab$.
b. Show that $\gcd(k, \ell) = 1$.
c. Suppose that $n \in \mathbb{Z}$ is such that $a \mid n$ and $b \mid n$. Show that $m \mid n$.
Because of parts (a) and (c) above, the number m is called the *least common multiple* of a and b and is written as $\text{lcm}(a, b)$. Since $dm = ab$ from part (a), it follows that $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$. Using this together with the Euclidean Algorithm, we can quickly compute least common multiples.

Problem 2: Let $\times$ be the cross product on $\mathbb{R}^3$.
a. Is $\times$ an associative operation on $\mathbb{R}^3$? Either prove or give an explicit counterexample.
b. Does $\times$ have an identity on $\mathbb{R}^3$? Prove your answer.

Problem 3: Consider the set $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ of nonnegative reals. Let $*$ be the binary operation on $\mathbb{R}^{\geq 0}$ given by exponentiation, i.e. $a * b = a^b$.
a. Is $*$ an associative operation on $\mathbb{R}^{\geq 0}$? Either prove or give an explicit counterexample.
b. Does $*$ have an identity on $\mathbb{R}^{\geq 0}$? Prove your answer.

Problem 4: Define a binary operation $*$ on $\mathbb{R}$ by letting $a * b = a + b + ab$.
a. Show that $*$ is commutative, i.e. that $a * b = b * a$ for all $a, b \in \mathbb{R}$.
b. Show that $*$ is associative, i.e. that $(a * b) * c = a * (b * c)$ for all $a, b, c \in \mathbb{R}$.
c. Show that $\mathbb{R}$ with operation $*$ has an identity element.
d. Show that the set of invertible elements of $*$ equals $\mathbb{R}\backslash\{-1\} = \{x \in \mathbb{R} : x \neq -1\}$.
Note: Using Corollary 4.3.5, it follows that $\mathbb{R}\backslash\{-1\}$ under $*$ and with the identity element from part (c) is an abelian group.

Problem 5: Let S be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

where $a \in \mathbb{R}$ and $a \neq 0$.
a. Show that if $A, B \in S$, then $AB \in S$. Thus, matrix multiplication is a binary operation on S.
b. Show that S with matrix multiplication has an identity element.
c. Notice that every matrix in S has determinant 0, so every matrix in S fails to be invertible in the linear algebra sense. Nevertheless, show that S is group under matrix multiplication with the identity from part (b).

Problem 6: Use the Euclidean Algorithm to show that $\overline{153} \in U(\mathbb{Z}/385\mathbb{Z})$ and to explicitly find its inverse.

Problem 7:
a. Write out the Cayley table of $U(\mathbb{Z}/18\mathbb{Z})$.
b. Compute (with explanation) the order of $\overline{11}$ in $U(\mathbb{Z}/18\mathbb{Z})$.

Problem 8: Suppose that $n \geq 3$. Let $\sigma \in S_n$ with $\sigma \neq id$. Show that there exists $\tau \in S_n$ such that $\sigma\tau \neq \tau\sigma$.
Hint: I strongly recommend that you avoid cycle notation and just work with σ as a function. Since $\sigma \neq id$, start by fixing an i with $\sigma(i) \neq i$. Now build a function $\tau \in S_n$ and show that it works.