Homework 13: Due Friday, December 13

Problem 1: Let R be an integral domain. Suppose that p is a prime element of R. Show that if $q \in R$ is an associate of p, then q is prime.

Problem 2: Show that if R is a UFD, then every irreducible element of R is prime.
Aside: Theorem 11.5.12 says that if R is an integral domain where $\|$ is well-founded, and every irreducible is prime, then R is a UFD. This problem is a partial converse.

Problem 3: Suppose that R is a PID. Let $a, b \in R$. Show that there exists a least common multiple of a and b. That is, show that there exists $c \in R$ with the following properties:

- $a \mid c$ and $b \mid c$.
- Whenever $d \in R$ satisfies both $a \mid d$ and $b \mid d$, it follows that $c \mid d$.

Hint: Think about the set of common multiples of a and b and how you can describe it as an ideal.

Problem 4: Give an example (with justification) of a field with 125 elements.

Problem 5: Let $p \in \mathbb{N}^+$ be prime.
a. Show that there exists a monic irreducible polynomial in $\mathbb{Z}/p\mathbb{Z}[x]$ of degree 2.
b. Show that there exists a field with p^2 elements.
Hint for a: You don't have to explicitly build one for each prime p. You just need to show one exists. How many monic polynomials of degree 2 are there?