Homework 11: Due Friday, November 22

Problem 1: Consider $\mathbb{R}$ and $\mathbb{C}$ as rings. Show that $\mathbb{R} \not\cong \mathbb{C}$.

Problem 2: Show that the only ideals of $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.

Problem 3: Let R be a ring and let I and J be ideals of R. Define the following set:

$$I + J = \{c + d : c \in I, d \in J\}.$$

Prove that $I + J$ is an ideal of R (it is the smallest ideal of R containing both I and J).

Problem 4: Consider the subring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of $\mathbb{R}$.
a. Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$.
b. Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

Problem 5 Let $p \in \mathbb{N}^+$ be prime. Consider the polynomial $f(x) = x^p - x$ in $\mathbb{Z}/p\mathbb{Z}[x]$. How many roots does $f(x)$ have in $\mathbb{Z}/p\mathbb{Z}$? Explain.

Problem 6: Working in the ring $\mathbb{Z}[x]$, let I be the ideal

$$I = \langle 2, x \rangle = \{p(x) \cdot 2 + q(x) \cdot x : p(x), q(x) \in \mathbb{Z}[x]\}.$$

Show that I is not a principal ideal in $\mathbb{Z}[x]$.

Problem 7: Recall from Homework 10 that an element $a \in R$ is *nilpotent* if there exists $n \in \mathbb{N}^+$ with $a^n = 0$. Let R be a commutative ring and let P be a prime ideal of R. Show that $a \in P$ for every nilpotent element $a \in R$.