Homework 4: Due Friday, September 25

Problem 1: Find the order of the following elements in the given direct product.
a. $((1\ 6\ 4)(3\ 7), (1\ 4\ 2\ 3)) \in S_9 \times S_4$
b. $(\overline{5}, \overline{7}, \overline{44}) \in \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/84\mathbb{Z}$
c. $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \overline{3}) \in GL_2(\mathbb{R}) \times U(\mathbb{Z}/13\mathbb{Z})$

Problem 2: Let G be group and let $a, g \in G$. The element gag^{-1} is called a *conjugate* of a.
a. Show that $(gag^{-1})^n = ga^ng^{-1}$ for all $n \in \mathbb{Z}$. You should start by giving a careful inductive argument for $n \in \mathbb{N}$.
b. Show that $|gag^{-1}| = |a|$. Thus, every conjugate of a has the same order as a.
c. Show that $|ab| = |ba|$ for all $a, b \in G$.

Problem 3: Let $n \in \mathbb{N}$ with $n \geq 2$.
a. Show that $\{(1\ \ a) : 2 \leq a \leq n\}$ generates S_n.
b. Show that $\{(a\ \ a+1) : 1 \leq a \leq n-1\}$ generates S_n.
c. Show that $\{(1\ \ 2), (1\ \ 2\ \ 3\ \cdots\ n)\}$ generates S_n.
Hint: Don't reinvent the wheel every time. You already know you can get everything from the transpositions. Once you've done part a, you know you can get everything from that smaller set, etc.

Problem 4: Let $n \in \mathbb{N}^+$.
a. Given $k \in \mathbb{N}$ with $2 \leq k \leq n$, find a formula for the number of k-cycles in S_n and explain why it is correct. (Remember that $(1\ 2\ 3) = (2\ 3\ 1)$ so don't count it twice.)
b. Find a formula for the number of permutation in S_n which are the product of two disjoint 2-cycles and explain why it is correct.
c. In the special case of $n = 5$, calculate the number of permutations of S_5 of each cycle type (so you should explicitly calculate the number of 4-cycles, the number of permutations which are the product of a 3-cycle and 2-cycle which are disjoint, etc.). Notice that all of your answers divide $|S_5| = 120$. This is not an accident, as we will see later.

Problem 5: Suppose that G and H are groups.
a. Show that if $G \times H$ is cyclic, then both G and H are cyclic.
b. Give a counterexample to the following statement: If G and H are both cyclic, then $G \times H$ is cyclic.
c. Suppose that G and H are both finite and cyclic. Assume also that $|G|$ and $|H|$ are relatively prime. Show that $G \times H$ is cyclic.

Problem 6: Let $n \geq 3$. Show that every element of A_n can be written as a product of 3-cycles (so the set of 3-cycles generates A_n).

Problem 7: This problem gives another interpretation of D_n as a subgroup of $GL_2(\mathbb{R})$ by thinking of rotation and flips as linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.
a. Let $\alpha, \beta \in \mathbb{R}$. Show that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

b. Let $n \geq 3$. Let

$$R = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \qquad\qquad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $|R| = n$, $|S| = 2$, and $SR = R^{-1}S$.