

## Homework 13: Due Friday, December 11

**Problem 1:** Let  $R$  be an integral domain. Suppose that  $p$  is a prime element of  $R$ . Show that if  $q \in R$  is an associate of  $p$ , then  $q$  is prime.

**Problem 2:** Show that if  $R$  is a UFD, then every irreducible element of  $R$  is prime.

*Aside:* Theorem 11.5.12 says that if  $R$  is an integral domain where  $\|$  is well-founded, and every irreducible is prime, then  $R$  is a UFD. This problem is a partial converse.

**Problem 3:** Suppose that  $R$  is a PID. Let  $a, b \in R$ . Show that there exists a least common multiple of  $a$  and  $b$ . That is, show that there exists  $c \in R$  with the following properties.

- $a \mid c$  and  $b \mid c$
- Whenever  $d \in R$  satisfies both  $a \mid d$  and  $b \mid d$ , it follows that  $c \mid d$ .

*Hint:* Think about the set of common multiples of  $a$  and  $b$  and how you can describe it as an ideal.

**Problem 4:** Give an example (with justification) of a field with 125 elements.

**Problem 5:** Let  $p \in \mathbb{N}^+$  be prime.

a. Show that there exists a monic irreducible polynomial in  $\mathbb{Z}/p\mathbb{Z}[x]$  of degree 2.

b. Show that there exists a field with  $p^2$  elements.

*Hint for a:* You don't have to explicitly build one for each prime  $p$ . You just need to show one exists. How many monic polynomials of degree 2 are there?