Homework 9 : Due Friday, September 17

Problem 1: Let $n \in \mathbb{N}^+$ and let

$$O(n, \mathbb{R}) = \{M \in GL_n(\mathbb{R}) : M^{-1} = M^T\}$$

Show that $O(n, \mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. This subgroup is called the *orthogonal group* of degree n. (Recall from linear algebra that an invertible $n \times n$ matrix M satisfies $M^{-1} = M^T$ if and only if the columns of M form an orthonormal basis of $\mathbb{R}^n$. These matrices are called *orthogonal matrices* and they give the distance-preserving linear transformations of $\mathbb{R}^n$.)

Problem 2: Let G be a group, and let H and K be subgroups of G.
a. Show that $H \cap K$ is a subgroup of G.
b. Show, by giving an explicit counterexample, that it need not be the case that $H \cup K$ is a subgroup of G.
Cultural Note: Since $SL_n(\mathbb{R})$ and $O(n, \mathbb{R})$ are both subgroups of $GL_n(\mathbb{R})$, it follows from the first part that their intersection is also a subgroup of $GL_n(\mathbb{R})$. This subgroup is denoted by $SO(n, \mathbb{R})$ and is called the *special orthogonal group* of degree n. Recall that an orthogonal matrix always has determinant ± 1, so $SO(n, \mathbb{R})$ simply throws out those elements of $O(n, \mathbb{R})$ which have determinant -1. The elements of $SO(n, \mathbb{R})$ give the linear transformations of $\mathbb{R}^n$ which are both distance-preserving and orientation-preserving.

Problem 3: Given a group G, we let

$$Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$$

That is, $Z(G)$ is the set of element of G which commute with *every* element of G. Show that $Z(G)$ is an abelian subgroup of G.

Problem 4: Let G be group and let $a, g \in G$. The element gag^{-1} is called a *conjugate* of a.
a. Show that $(gag^{-1})^n = ga^n g^{-1}$ for all $n \in \mathbb{Z}$. You should start by giving a careful inductive argument for $n \in \mathbb{N}$.
b. Show that $|gag^{-1}| = |a|$. Thus, every conjugate of a has the same order as a.
c. Show that $|ab| = |ba|$ for all $a, b \in G$.

Problem 5: Let G be a group with even order. Show that G has an element of order 2.
Hint: You need to show that there is a nonidentity elements which is its own inverse. Think about taking the inverse of each element and see what happens.