

## Homework 7 : Due Monday, September 13

**Problem 1:** Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.

- a.  $A = \mathbb{Z}$  where  $a \sim b$  means  $a - b \neq 1$ .
- b.  $A = \mathbb{Z}$  where  $a \sim b$  means that both  $a$  and  $b$  are even.
- c.  $A = \mathbb{Z}$  where  $a \sim b$  means  $a \mid b$ .
- d.  $A = \mathbb{R}$  where  $a \sim b$  means  $|a - b| \leq 10$ .
- e.  $A = \mathbb{N} \times \mathbb{N}$  where  $(a, b) \sim (c, d)$  means  $a + d = b + c$ .

**Problem 2:** As in class, let  $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$  and let  $\sim$  be the equivalence relation given by  $(a, b) \sim (c, d)$  if  $ad = bc$ . Let  $Q$  be the set of equivalence classes of  $A$  under  $\sim$ .

- a. Show that if  $(a_1, b_1) \sim (c_1, d_1)$  and  $(a_2, b_2) \sim (c_2, d_2)$ , then  $(a_1 a_2, b_1 b_2) \sim (c_1 c_2, d_1 d_2)$ .
- b. Show that if  $(a_1, b_1) \sim (c_1, d_1)$  and  $(a_2, b_2) \sim (c_2, d_2)$ , then  $(a_1 b_2 + a_2 b_1, b_1 b_2) \sim (c_1 d_2 + c_2 d_1, d_1 d_2)$ .

The above parts show that the operations of addition and multiplication of fractions you learned in grade school are indeed well-defined on  $Q$ . In other words, the definitions

$$\overline{(a, b)} \cdot \overline{(c, d)} = \overline{(ac, bd)} \quad \overline{(a, b)} + \overline{(c, d)} = \overline{(ad + bc, bd)}$$

make sense.

- c. Let  $a, b \in \mathbb{Z}$  with  $b \neq 0$ . Show that  $\overline{(a, b)} + \overline{(-a, b)} = \overline{(0, 1)}$ .
- d. Let  $a, b \in \mathbb{Z}$  with both  $a, b \neq 0$ . Show that  $\overline{(a, b)} \cdot \overline{(b, a)} = \overline{(1, 1)}$ .

Thus, every element has an additive inverse and every nonzero element (i.e. every element other than  $\overline{(0, 1)}$ ) has a multiplicative inverse.

**Problem 3:** Let  $Q$  be as in Problem 2 and let  $P$  be the projective line, i.e.  $P$  is the set of equivalence classes of the set  $\mathbb{R}^2 \setminus \{(0, 0)\}$  under the equivalence relation  $(x_1, y_1) \sim (x_2, y_2)$  if there exists a real number  $\lambda \neq 0$  with  $(x_2, y_2) = (\lambda x_1, \lambda y_1)$ . Determine which of the following functions on equivalence classes are well-defined. In each case, either give a proof or a specific counterexample.

- a.  $f: Q \rightarrow \mathbb{Z}$  defined by  $f(\overline{(a, b)}) = a - b$ .
- b.  $f: Q \rightarrow Q$  defined by  $f(\overline{(a, b)}) = \overline{(a^2 + 3ab + b^2, 5b^2)}$ .
- c.  $f: P \rightarrow \mathbb{R}$  defined by

$$f(\overline{(x, y)}) = \frac{2xy^3 + 5xy}{x^4 + y^4}$$

- d.  $f: P \rightarrow P$  defined by  $f(\overline{(x, y)}) = \overline{(x^3 + 5xy^2, y^3)}$ .

**Problem 4:** A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

“If  $a \sim b$ , then  $b \sim a$  by symmetry, so  $a \sim a$  by transitivity. This gives the reflexive property.”

Now you know that their argument must be wrong because one of the examples in Problem 1 is symmetric and transitive but not reflexive. Pinpoint the error in your friend’s argument. Be as explicit and descriptive as you can.