Homework 6 : Due Friday, September 10

Problem 1: Show that there are arbitrarily large gaps in the primes, i.e. that for every $n \in \mathbb{N}^+$, there exist n consecutive composite numbers.
Hint: $n!$ is your friend.

Problem 2: Show that if $a, b \in \mathbb{N}^+$ satisfy $a \mid b$, then $(2^a - 1) \mid (2^b - 1)$. Conclude that if $n \in \mathbb{N}^+$ is such that $2^n - 1$ is prime, then n is prime. Primes of the form $2^p - 1$ for a prime p are called *Mersenne primes*. It is an open problem whether there are infinitely many of them.

Problem 3: An integer $n \in \mathbb{Z}$ is a *square* if there exists $m \in \mathbb{Z}$ with $n = m^2$. Suppose that $a, b \in \mathbb{N}^+$ are relatively prime and that ab is a square. Show that both a and b are squares.

Problem 4: Let $S = \{2n : n \in \mathbb{Z}\}$ be the set of even integers. Notice that the sum and product of two elements of S is still an element of S. Call an element of $a \in S$ *irreducible* if $a > 0$ and there is no way to write $a = bc$ with $b, c \in S$. Notice that 6 is irreducible in S even though it is not prime in $\mathbb{Z}$ (although $6 = 2 \cdot 3$, we have that $3 \notin S$).
a. Give a characterization of the irreducible elements of S.
b. Show that the analogue of Fundamental Theorem of Arithmetic fails in S by finding a positive element of S which does *not* factor uniquely (up to order) into irreducibles.