Homework 30 : Due Friday, December 10

Problem 1: Give an example (with justification) of a field with 32 elements.

Problem 2: Determine (with proof) the minimal polynomial of $\sqrt{2-\sqrt{2}}$ over $\mathbb{Q}$.

Problem 3: Show that $x^4 - 10x^2 + 1$ is the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

Problem 4: Let $p(x) = x^3 + 3x + 2$.
a. Show that $p(x)$ is irreducible in $\mathbb{Q}[x]$.
b. Show that $p(x)$ has exactly one root in $\mathbb{R}$.
c. Let α be the unique real root of $p(x)$. We know from part a that $p(x)$ is the minimal polynomial of α over $\mathbb{R}$, hence $\{1, \alpha, \alpha^2\}$ is a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ by Theorem 12.15. In particular, we have

$$\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$$

Now we clearly have $\alpha^5 - 2\alpha^3 + 42 \in \mathbb{Q}(\alpha)$. Find $a, b, c \in \mathbb{Q}$ such that

$$\alpha^5 - 2\alpha^3 + 42 = a + b\alpha + c\alpha^2$$