Homework 26 : Due Friday, November 19

Problem 1: Given $f(x), g(x) \in F[x]$ in each of the following cases, calculate $q(x), r(x) \in F[x]$ with $f(x) = q(x)g(x) + r(x)$ and either $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.
a. $F = \mathbb{Z}/2\mathbb{Z}$, $f(x) = x^5 + x^3 + x^2 + \overline{1}$ and $g(x) = x^2 + x$.
b. $F = \mathbb{Z}/5\mathbb{Z}$, $f(x) = x^3 + \overline{3}x^2 + \overline{2}$ and $g(x) = \overline{4}x^2 + \overline{1}$

Problem 2: Find a nonconstant polynomial in $\mathbb{Z}/4\mathbb{Z}$ which is a unit. Moreover, show that for every $n \in \mathbb{N}^+$, there exists a polynomial in $\mathbb{Z}/4\mathbb{Z}[x]$ of degree n which is a unit.

Problem 3: Give an example of a nonzero polynomial $f(x) \in \mathbb{Z}/6\mathbb{Z}[x]$ which has more than $\deg(f(x))$ many roots.

Problem 4:
a. Let F be a field. Suppose that $f(x) \in F[x]$ is a polynomial of degree either 2 or 3. Show that $f(x)$ is irreducible in $F[x]$ if and only if $f(x)$ has no root in F.
b. Given an example of a degree 4 polynomial in $\mathbb{Q}[x]$ which has no root in $\mathbb{Q}$ but is *not* irreducible in $\mathbb{Q}[x]$.

Problem 5: Let F be a field. Show that every nonconstant polynomial in $F[x]$ can be written as a product of irreducible polynomials in $F[x]$. Give a careful proof.
Hint: This is the analogue of the statement that every integer can be written as a product of primes. It might help to go back and look at the proof.