Homework 25 : Due Wednesday, November 17

Problem 1: Let R be a commutative ring and let $a, b \in R$.
a. Show that $a \mid b$ if and only if $\langle b \rangle \subseteq \langle a \rangle$.
b. Let R be an integral domain. Show that $\langle a \rangle = \langle b \rangle$ if and only if a and b are associates.

Problem 2: Let R be an integral domain and let $p \in R$.
a. Show that if p is irreducible, then every associate of p is irreducible.
b. Show that if p is prime, then every associate of p is prime.

Problem 3: Let R be a commutative ring and let I and J be ideals of R. The product of I and J, denoted IJ, is the set

$$IJ = \{c_1d_1 + c_2d_2 + \cdots + c_kd_k : k \in \mathbb{N}^+, c_i \in I, d_i \in J\}$$

That is, elements of IJ are the finite sums of elements which are formed as the product of an element of I with an element of J.
a. Prove that IJ is an ideal of R.
b. Show that $IJ \subseteq I \cap J$.
c. Show that if $I = \langle a \rangle$ and $J = \langle b \rangle$, then $IJ = \langle ab \rangle$.
d. Find an example of ideals I and J of some commutative ring R for which $IJ \subsetneq I \cap J$.

Problem 4: Working in the ring $\mathbb{Z}[x]$, let I be the ideal

$$I = \langle 2, x \rangle = \{p(x) \cdot 2 + q(x) \cdot x : p(x), q(x) \in \mathbb{Z}[x]\}$$

Show that I is not a principal ideal in $\mathbb{Z}[x]$.