Homework 24 : Due Monday, November 15

Problem 1: Let $R = \mathbb{Z}/20\mathbb{Z}$ and let $I = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\}$. It is straightforward to check that I is an ideal of R (you can verify this directly, but in the Correspondence Theorem, I corresponds to the ideal $4\mathbb{Z}$ of $\mathbb{Z}$ since $20\mathbb{Z} \subseteq 4\mathbb{Z}$). Write out the cosets of I in R along with addition and multiplication tables for the ring R/I.

Problem 2: Let R be a ring and let I and J be ideals of R. Define the following set.

$$I + J = \{c + d : c \in I, d \in J\}$$

a. Prove that $I + J$ is an ideal of R (it is the smallest ideal of R containing both I and J).
b. We know that every ideal of $\mathbb{Z}$ equals $\langle n \rangle$ for some $n \in \mathbb{N}^+$. Show that for any $m, n \in \mathbb{Z}$, we have $\langle m \rangle + \langle n \rangle = \langle d \rangle$ where $d = \gcd(m, n)$.

Problem 3: Recall from Homework 22 that $a \in R$ is *nilpotent* if there exists $n \in \mathbb{N}^+$ with $a^n = 0$. Let R be a commutative ring and let P be a prime ideal of R. Show that $a \in P$ for every nilpotent element $a \in R$.

Problem 4: Recall from Homework 22 that a Boolean ring is a ring R for which $a^2 = a$ for all $a \in R$. You proved every Boolean ring is commutative.
a. Show that if R is both a Boolean ring and an integral domain, then $R \cong \mathbb{Z}/2\mathbb{Z}$.
b. Show that if R is a Boolean ring and I is an ideal of R, then R/I is a Boolean ring.
c. Show that every prime ideal in a Boolean ring is a maximal ideal.

Problem 5: Let $C[0, 1]$ be set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Define $+$ and $\cdot$ on R to be the usual addition and multiplication of functions. That is, we define

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} (f \cdot g)(x) = f(x) \cdot g(x)$$

With these operations, the set $C[0, 1]$ is a ring (the additive identity is the constant function 0, and the multiplicative identity is the constant function 1). Let

$$I = \{f \in C[0, 1] : f(0) = 0 = f(1)\}$$

a. Show that I is an ideal of R.
b. Show that I is not a prime ideal of R.