Homework 23 : Due Friday, November 12

Problem 1: Consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ as a direct product (so addition and multiplication are componentwise). Determine, with explanation, which of the following subsets are ideals of R.
a. $\{(a, 0) : a \in \mathbb{Z}\}$
b. $\{(a, a) : a \in \mathbb{Z}\}$
c. $\{(2a, 3b) : a, b \in \mathbb{Z}\}$

Problem 2: Define $\varphi \colon \mathbb{C} \to M_2(\mathbb{R})$ by letting

$$\varphi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Show that φ is an injective ring homomorphism (so $\mathbb{C}$ is isomorphic to the subring range(φ) of $M_2(\mathbb{R})$).

Problem 3: Suppose that R and S are rings and that $\varphi \colon R \to S$ is a function such that

- $\varphi(r + s) = \varphi(r) + \varphi(s)$ for all $r, s \in R$.
- $\varphi(rs) = \varphi(r) \cdot \varphi(s)$ for all $r, s \in R$.

a. Show that $\varphi(1_R)$ is an idempotent of S.
b. Show that if φ is surjective, then $\varphi(1_R) = 1_S$.
c. Suppose that S is an integral domain and that φ is not the zero function (i.e. there exists $r \in R$ with $\varphi(r) \neq 0_S$). Show that $\varphi(1_R) = 1_S$.

Problem 4: Consider $\mathbb{R}$ and $\mathbb{C}$ as rings. Show that $\mathbb{R} \not\cong \mathbb{C}$.

Problem 5: Show that the only ideals of $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.
Hint: Suppose that $I \neq \{0\}$. First show that I contains an invertible matrix.