Homework 22 : Due Wednesday, November 10

Problem 1: An element $e \in R$ is called an *idempotent* if $e^2 = e$. Notice that 0 and 1 are idempotents in every ring R.
a. Show that if $e \in R$ is both a unit and an idempotent, then $e = 1$.
b. Show that if R is an integral domain, then 0 and 1 are the only idempotents of R.
c. Find all idempotents in $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/18\mathbb{Z}$.

Problem 2: An element $a \in R$ is called a *nilpotent* element if there exists $n \in \mathbb{N}^+$ with $a^n = 0$.
a. Show that every nonzero nilpotent element is a zero divisor.
b. Show that if a is both nilpotent and an idempotent, then $a = 0$.
c. Show that if a is nilpotent, then $1 - a$ is a unit.
d. Given $n \in \mathbb{N}^+$, describe all nilpotent elements in $\mathbb{Z}/n\mathbb{Z}$. *Hint:* Start with the prime factorization of n.

Problem 3: Let X be a nonempty set. Let $R = \mathcal{P}(X)$ be the power set of X, i.e. the set of all subsets of X. We define $+$ and $\cdot$ on elements of R as follows. Given $A, B \in \mathcal{P}(X)$, define

$$A + B = A \cup B$$

and let

$$A \cdot B = A \cap B$$

a. Show that with these operations, R is *not* a ring.

Let's scrap the above operations and try again. Given two sets A and B, the symmetric difference of A and B, denoted $A \triangle B$, is

$$A \triangle B = (A \backslash B) \cup (B \backslash A)$$

i.e. $A \triangle B$ is the set of elements in exactly one of A and B. Now define $+$ and $\cdot$ on elements of R as follows. Given $A, B \in \mathcal{P}(X)$, let

$$A + B = A \triangle B$$

and let

$$A \cdot B = A \cap B$$

It turns out that with these operations, R is a commutative ring, although some of the axioms are a pain to check (especially associatively of $+$ and distributivity).

b. Explain what the additive identity and multiplicative identity are in this ring, and explain what the additive inverse of an element is.

Problem 4: A Boolean ring is a ring in which every element is an idempotent, i.e. $a^2 = a$ for all $a \in R$. For example, $\mathbb{Z}/2\mathbb{Z}$ is a Boolean ring, as are all of the examples in Problem 3b.
a. Show that if R is a Boolean ring, then $a + a = 0$ for all $a \in R$.
b. Show that every Boolean ring is commutative.

Problem 5: Recall that $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. Define a function $N \colon \mathbb{Z}[i] \to \mathbb{N}$ by letting $N(a + bi) = a^2 + b^2$. The function N is called the *norm* on $\mathbb{Z}[i]$.
a. Show that $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[i]$.
b. Show that if α is a unit in $\mathbb{Z}[i]$, then $N(\alpha) = 1$.
c. Find all units in $\mathbb{Z}[i]$.