

Homework 21 : Due Monday, November 1

Problem 1: Compute, with explanation, the conjugacy classes and the Class Equation for each of the following groups.

- S_4
- A_4
- D_5

Note: You can really cut down on computations using the ideas from class. For part c, it is even possible to compute the Class Equation first and use it along with previous homework to do very few computations.

Problem 2: Find all finite groups which have exactly two conjugacy classes.

Problem 3: Suppose that G is a nonabelian group with $|G| = 125$. Show that $|Z(G)| = 5$ and that $G/Z(G) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 4: This problem provides another proof of Cauchy's Theorem. Let G be a group and suppose that p is a prime which divides $|G|$. Let

$$X = \{(a_1, a_2, \dots, a_{p-1}, a_p) \in G^p : a_1 a_2 \cdots a_{p-1} a_p = e\}$$

i.e. X consists of all p -tuples of elements of G such that when you multiply them in the given order you get the identity.

- Give four examples of elements of X in the special case when $G = S_3$ and $p = 3$.
- Show that $|X| = |G|^{p-1}$.
- Show that if $(a_1, a_2, \dots, a_{p-1}, a_p) \in X$, then $(a_2, a_3, \dots, a_p, a_1) \in X$. It follows that any cyclic shift of an element of X remains in X .

Let H be the subgroup of S_p generated by the element $(12 \dots p)$, so $|H| = p$. Let H act on X by permuting the elements, i.e. if $\sigma \in H$ and $(a_1, a_2, \dots, a_{p-1}, a_p) \in X$, then

$$\sigma * (a_1, a_2, \dots, a_{p-1}, a_p) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(p-1)}, a_{\sigma(p)})$$

In other words, $(1 2 \dots p)$ shifts an element in X to the left one (as in part b), $(1 2 \dots p)^2$ shifts to the left 2, etc. It is straightforward to check that this is an action of H on X .

- Notice that $|\mathcal{O}_{(e, e, \dots, e, e)}| = 1$. Show that there exists at least one other orbit of size 1.
- Conclude that G has an element of order p .