Homework 20 : Due Friday, October 29

Problem 1: Let $GL_n(\mathbb{R})$ act on $\mathbb{R}^n$ as usual, so $A * \mathbf{x} = A\mathbf{x}$.
a. Find (with proof) the orbits of the action of the subgroup $SL_2(\mathbb{R})$ on $\mathbb{R}^2$.
b. Find (with proof) the orbits of the action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$.
Hint: For part b especially, it is possible to build lots of matrices to make things work, but you can by with far less effort if you use some theory.

Problem 2: Suppose that G acts on X. We saw in class that for each $a \in G$, the function $\pi_a \colon X \to X$ defined by $\pi_a(x) = a * x$ is a permutation of X. Define $\varphi \colon G \to S_X$ by letting $\varphi(a) = \pi_a$. Show that φ is a homomorphism.

Problem 3: Suppose that G acts on X. Let $H = \{a \in G : a * x = x \text{ for all } x \in X\}$. Show that H is a normal subgroup of G. H is called the *kernel* of the action. Notice that H is in the intersection of all the stabilizers G_x.

Problem 4: Let $G = \mathbb{R}$ (under addition) and let $X = \mathbb{R}^2$. Define a function from $G \times X$ to X by $a * (x, y) = (x + ay, y)$.
a. Show that $*$ is a action of G on X.
b. Describe the orbits of the action geometrically. Be careful!
c. Describe the stabilizers of each point.

Problem 5: Let $G = S_3$ and let

$$X = \{1, 2, 3\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Define a function from $G \times X$ to X by $\sigma * (x, y) = (\sigma(x), \sigma(y))$.
a. Show that $*$ is a action of G on X.
b. Find the orbits and stabilizers of each point.