Homework 2 : Due Wednesday, September 1

Problem 1: Define a binary operation $*$ on $\mathbb{R}$ by letting $a * b = a + b + ab$. Let

$$G = \mathbb{R}\backslash\{-1\} = \{x \in \mathbb{R} : x \neq -1\}$$

a. Show that $*$ is commutative, i.e. that $a * b = b * a$ for all $a, b \in \mathbb{R}$.
b. Show that if $a \in G$ and $b \in G$, then $a * b \in G$. Thus, $*$ is a binary operation on G.
c. Show that $\mathbb{R}$ with operation $*$ has an identity element.
d. Letting e be the identity element found in part c, show that $(G, *, e)$ is a group.

Problem 2: Let S be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

where $a \in \mathbb{R}$ and $a \neq 0$.
a. Show that if $A, B \in S$, then $AB \in S$.
b. Show that S with matrix multiplication has an identity element.
c. Notice that every matrix in S has determinant 0, so every matrix in S fails to be invertible (in the linear algebra sense). Nevertheless, show that S is group under matrix multiplication with the identity from part b.

Problem 3: Let G be a group. Suppose that $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ for all $a, b \in G$. Show that G is abelian.