Homework 19 : Due Wednesday, October 27

Problem 1: Let $G = (\mathbb{R}, +)$ and let $H = (\mathbb{R}\backslash\{0\}, \cdot)$. Show that $G \ncong H$.

Problem 2: Let G and H be groups and let $\varphi\colon G \to H$ and $\psi\colon G \to H$ be homomorphisms. Show that $\{g \in G : \varphi(g) = \psi(g)\}$ is a subgroup of G.
Note: It follows that if $G = \langle c \rangle$, and $\varphi(c) = \psi(c)$, then $\varphi = \psi$ (because the smallest subgroup of G containing c is all of G). Similarly, if $A \subseteq G$ is a subset which generates G, and $\varphi(a) = \psi(a)$ for all $a \in A$, then $\varphi = \psi$.

Problem 3: Let p and q be distinct primes. Suppose that G is an *abelian* group of order pq. Show that G is the internal direct product of two nontrivial subgroups of G and use it to conclude that $G \cong \mathbb{Z}/pq\mathbb{Z}$. Thus, up to isomorphism, there is only one abelian group of order pq.
Hint: Start with Theorem 7.19.

Problem 4: Given a group G, consider the group $G \times G$ and the subset $D = \{(a, a) : a \in G\}$. It is straightforward to check that D is a subgroup of $G \times G$ and that $D \cong G$.
a. Show that if $G = S_3$, then D is not a normal subgroup of $G \times G$.
b. Suppose that G is abelian. Find a surjective homomorphism $\varphi\colon G \times G \to G$ with $\ker(\varphi) = D$ and use it to conclude that $(G \times G)/D \cong G$.

Problem 5: Let $G = \mathbb{Z}/24\mathbb{Z}$, let $H = \langle \overline{4} \rangle$ and let $K = \langle \overline{6} \rangle$. The Second Isomorphism Theorems says that

$$\frac{H}{H \cap N} \cong \frac{H + N}{N}$$

(note that we wrote $H + N$ rather than HN because the group operation is addition).
a. Explicitly calculate $H \cap N$ and $H + N$.
b. Explicitly list the cosets in the groups $H/(H \cap N)$ and $(H + N)/N$.
c. Follow the proof of the Second Isomorphism Theorem to explicitly write down the isomorphism from $H/(H \cap N)$ to $(H + N)/N$ using your descriptions of the elements in part b.