Homework 18 : Due Friday, October 15

Problem 1: Suppose that $G_1 \cong H_1$ and $G_2 \cong H_2$. Show that $G_1 \times G_2 \cong H_1 \times H_2$.

Problem 2: Show that S_n is isomorphic to a subgroup of A_{n+2}.

Problem 3: Consider the group $G = U(\mathbb{Z}/15\mathbb{Z})$. Find cyclic subgroups H and K of G such that G is the internal direct product of H and K. Use this to find $m, n \in \mathbb{N}^+$ such that $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Problem 4: Let G be a group of order 4. Show that either $G \cong \mathbb{Z}/4\mathbb{Z}$ or $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, up to isomorphism, there are exactly two groups of order 4.

Problem 5: An *automorphism* of a group G is an isomorphism $\varphi\colon G \to G$.
a. Let G be a group, and fix $g \in G$. Define a function $\varphi_g\colon G \to G$ by letting $\varphi_g(a) = gag^{-1}$. Show that φ_g is an automorphism of G.
b. Suppose that G is a cyclic group or order $n \in \mathbb{N}^+$. Let $k \in \mathbb{Z}$ with $\gcd(k, n) = 1$. Define $\psi\colon G \to G$ by letting $\psi(a) = a^k$. Show that ψ is an automorphism of G. Furthermore, show that if $k \not\equiv_n 1$, then $\psi \neq \varphi_g$ for every $g \in G$.