Homework 16 : Due Monday, October 11

Problem 1: Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group with the property that every element has finite order.

Problem 2: Let G be a group and let H be a normal subgroup of G.
a. Show that G/H is abelian if and only if $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.
b. Suppose $[G : H]$ is finite and let $m = [G : H]$. Show that $a^m \in H$ for all $a \in G$.

Problem 3: Suppose that G is a group with $|G| \neq 1$ and $|G|$ not prime (so either $|G|$ is composite and greater than 1, or $|G| = \infty$). Show that there exists a subgroup H of G with $H \neq \{e\}$ and $H \neq G$.
b. Show that the only abelian simple groups are the cyclic groups of prime order.

Problem 4:
a. Suppose that G is a group with the property that $G/Z(G)$ is cyclic. Show that G is abelian.
b. Suppose that G is a group and $|G| = pq$ where p and q are (not necessarily distinct) primes. Show that either $Z(G) = \{e\}$ or $Z(G) = G$.