Homework 15 : Due Friday, October 8

Problem 1: For each of the following subgroups H of the given group G, determine if H is a normal subgroup of G.
a. $G = D_4$ and $H = \{id, s\}$.
b. $G = D_4$ and $H = \{id, rs\}$.
c. $G = A_4$ and $H = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. (First check that H is indeed a subgroup of G).
Suggestion: Normal subgroups have many equivalent characterizations. In each part, pick one of these which will make your life easy.

Problem 2: Suppose that H and K are both normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.

Problem 3: Suppose that G is a group and H is subgroup of G. Show that if $[G : H] = 2$, then H is a normal subgroup of G.

Problem 4: If G is a group and H and K are subgroups of G, we define

$$HK = \{hk : h \in H, k \in K\}$$

a. Consider $G = S_3$, $H = \langle (1\ 2) \rangle$ and $K = \langle (1\ 3) \rangle$. Show that HK is *not* a subgroup of G.
b. Suppose that H is a subgroup of G and N is a *normal* subgroup of G. Show that HN is a subgroup of G. (Note: Do not assume that H is a normal subgroup of G).

Problem 5:
a. Let G be a group and let H be a subgroup of G. Let $g \in G$. Show that the set gHg^{-1} is a subgroup of G and that $|gHg^{-1}| = |H|$.
b. Let G be a group. Suppose that $k \in \mathbb{N}^+$ is such that G has a unique subgroup of order k. If H is the unique subgroup of G of order k, show that H is a normal subgroup of G.