Homework 14 : Due Wednesday, October 6

Problem 1: Find the order of the following elements in the given direct product.
a. $(\overline{5}, \overline{7}, \overline{44}) \in \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/84\mathbb{Z}$
b. $((1\ 6\ 4)(3\ 7), r^{14}) \in S_9 \times D_{20}$
c. $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \overline{3}) \in O(2, \mathbb{R}) \times U(\mathbb{Z}/13\mathbb{Z})$

Problem 2: Suppose that $p \in \mathbb{N}^+$ is an odd prime and that there exists $a \in \mathbb{Z}$ with $a^2 \equiv_p -1$. Show that $p \equiv_4 1$.
Note: The converse statement is also true (if p is prime and $p \equiv_4 1$, then there exists $a \in \mathbb{Z}$ with $a^2 \equiv_p -1$), but this requires some more advanced number theory.

Problem 3: Suppose that H is a subgroup of D_n and that $|H|$ is odd. Show that H is cyclic.

Problem 4: Suppose that H is a subgroup of a group G with $[G : H] = 2$. Suppose that $a, b \in G$ with both $a \notin H$ and $b \notin H$. Show that $ab \in H$.
Hint: Think about the four cosets eH, aH, bH, and abH.

Problem 5: Suppose that H and G are groups.
a. Show that if $G \times H$ is cyclic, then both G and H are cyclic.
b. Suppose that G and H are both finite and cyclic. Show that $G \times H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.