Homework 1 : Due Monday, August 30

Problem 1: Let $\times$ be the cross product on $\mathbb{R}^3$.
a. Is $\times$ an associative operation on $\mathbb{R}^3$? Explain.
b. Does $\times$ have an identity on $\mathbb{R}^3$? Explain.

Problem 2: Consider the set $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ of nonnegative reals. Let $*$ be the binary operation on $\mathbb{R}^{\geq 0}$ given by exponentiation, i.e. $a * b = a^b$.
a. Is $*$ an associative operation on $\mathbb{R}^{\geq 0}$? Explain.
b. Does $*$ have an identity on $\mathbb{R}^{\geq 0}$? Explain.

Problem 3: Let $(G, \cdot, e)$ be a group and let $a, b, c, d \in G$. Show that

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot (b \cdot c)) \cdot d$$

Problem 4: Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. Now we know from linear algebra that matrix multiplication is associative and that the identity matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so the identity matrix is in G.
a. Show that if $A, B \in G$, then $AB \in G$. Thus, matrix multiplication is indeed a binary operation on G.
b. Show that if $A \in G$, then A is an invertible matrix and moreover $A^{-1} \in G$. Thus, G is a group under matrix multiplication. (Notice that if some matrix $A \in G$ was invertible as a matrix but its inverse was not in G, then G would *not* be a group.)