Homework 29 : Due Wednesday, December 9

Problem 1: Let R be an integral domain.
a. Define a relation $\sim$ by $a \sim b$ if there exists a unit $u \in R$ with $a = bu$ (this is our definition of associates). Show that $\sim$ is an equivalence relation.
b. Show that if a is an irreducible and b is an associate of a, then b is irreducible.

Problem 2: Let R be a PID and let $a, b \in R$. The ideal

$$\langle a, b\rangle = \{ra + sb : r, s \in R\}$$

must be principal. Suppose that $d \in R$ satisfies $\langle d\rangle = \langle a, b\rangle$.
a. Show that $d \mid a$ and $d \mid b$.
b. Show that if $c \in R$ is such that $c \mid a$ and $c \mid b$, then $c \mid d$.
c. Show that if d' is any other element of R satisfying both parts a and b, then d and d' are associates.
Taken together, this problem shows that the concept of *gcd* makes sense in any PID (up to associates).

Problem 3: Let D be a positive integer which is not divisible by the square of a prime (such integers are called *squarefree*). Let

$$R = \mathbb{Z}[\sqrt{D} \cdot i] = \{a + b\sqrt{D} \cdot i \in \mathbb{C} : a, b \in \mathbb{Z}\}$$

Define $N \colon R \to \mathbb{N} \cup \{0\}$ by

$$N(a + b\sqrt{D} \cdot i) = a^2 + Db^2$$

a. Show that $N(xy) = N(x) \cdot N(y)$ for all $x, y \in R$.
b. Show that $x \in R$ is a unit if and only if $N(x) = 1$.
c. Find all of the units in each $\mathbb{Z}[\sqrt{D} \cdot i]$.
d. Show that when $D = 5$, then 21 does not factor uniquely as a product of irreducibles. Explain why this shows that $\mathbb{Z}[\sqrt{D} \cdot i]$ is not a PID.
e. In the Gaussian integers (i.e. when $D = 1$), show that 3 and 11 are irreducible, but 5 and 13 are not.

Problem 4: A commutative ring R is Artinian if for every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

there exists an N such that $I_k = I_N$ for all $k \geq N$. Show that if R is an Artinian integral domain, then R is a field. (It is a nontrivial theorem that every Artinian commutative ring with identity is Noetherian.)