Homework 24 : Due Monday, November 23

Problem 1: Given a prime p, let

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \ : \ a, b \in \mathbb{Z} \text{ and } gcd(b, p) = 1\}$$

Show that $\mathbb{Z}_{(p)}$ is a subring of $\mathbb{Q}$. $\mathbb{Z}_p$ is called the ring of integers localized at p.

Problem 2: Given two sets A and B, the symmetric difference of A and B, denoted $A \triangle B$, is

$$A \triangle B = (A \backslash B) \cup (B \backslash A)$$

i.e. $A \triangle B$ is the set of elements in exactly one of A and B.

Let X be a set. Let $R = \mathcal{P}(X)$ be the power set of X, i.e. the set of all subsets of X. We define $+$ on $\cdot$ on elements of R as follows. Given $A, B \in \mathcal{P}(X)$, let

$$A + B = A \triangle B$$

and let

$$A \cdot B = A \cap B$$

Show that with these operations, R is a commutative ring with identity.

Problem 3: A Boolean ring is a ring for which $a^2 = a$ for all $a \in R$. For example, $\mathbb{Z}_2$ is a Boolean ring, as are all of the rings from Problem 2.

a. Show that if R is a Boolean ring, then $a + a = 0$ for all $a \in R$.

b. Show that every Boolean ring is commutative.

c. Show that if a Boolean ring R is an integral domain, then $R \cong \mathbb{Z}_2$.

Problem 4: Let R be an integral domain and let $a, b \in R$. Show that $\langle a \rangle = \langle b \rangle$ if and only if there exists a unit $u \in R$ with $a = bu$.

Problem 5: Let $R = C[0, 1]$ be the ring of all continuous functions on $[0, 1]$. Let

$$I = \{f \in C[0, 1] : f(0) = 0 = f(1)\}$$

a. Show that I is an ideal of R.

b. Show that I is not a prime ideal of R.